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THE GOODNESS OF $\{S,a\}$-EOL FORMS IS DECIDABLE

G. ROZENBERG
Institute of Applied Mathematics and Computer Science, University of Leiden, Leiden, The Netherlands

R. VERRAEDT*
Department of Mathematics, University of Antwerp, U.I.A., B-2610 Wilrijk, Belgium

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It is proved that given an EOL form F with one nonterminal and one terminal (a so-called $\{S,a\}$-EOL form) it is decidable whether or not F is good. As a corollary we prove that for a good $\{S,a\}$-EOL form F such that $L(F)\neq\emptyset$ either F is vomplete or $L(F)=\{a\}$.

Introduction

Decidability problems constitute one of the central topics in formal language theory. In the theory of grammatical similarity, in particular the theory of EOL forms, a lot of decidability questions arise. Some of them are proved to be decidable, e.g., m-completeness for some subclass of EOL forms (see [2] and [3]), simulations of one EOL form by another EOL form (see [6], [7] and [8]), vompleteness of EOL forms (see [17]). Other problems turn out to be undecidable, e.g., the x-form equivalence problem where x is the full uniform interpretation mechanism (see [11]) or the nonterminal fixed mechanism (see [16]). By now many problems are open or only partially solved, e.g., the completeness problem (see [5]) and the form equivalence problem (see [10]) for EOL forms (under the ordinary interpretation mechanism as defined in [10]).

In this paper we attack the following problem: for an EOL form F is it decidable whether or not F is good. Goodness of EOL forms is first defined in [12] and further studied in [13]. In [12] also examples of good and bad EOL forms are given. A lot of conditions on EOL forms which imply badness are given in the literature (see [1], [4], [13], [15] and [18]). We will concentrate on $\{S,a\}$-EOL forms, i.e., EOL forms with one nonterminal and one terminal.

The paper is organized as follows. In Section 1 we recall some basics of the theory of EOL systems and forms and give some basic conditions which must hold in a good EOL form. In the following sections we study several subclasses of $\{S,a\}$-EOL forms separately, i.e., terminal propagating forms in Section 2, terminal erasing

*Research assistant of the Belgian National Fund of Scientific Research (NFWO).

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initial forcing free forms in Section 3 and terminal erasing initial forcing forms in Section 4. Finally in Section 5 we prove that given an $\{S, a\}$-EOL form F it is decidable whether or not F is good. As a corollary we also prove that for a good $\{S, a\}$-EOL form F such that $L(F) \neq \emptyset$ either F is vomplete or $L(F) = \{a\}$.

1. Preliminaries and basic results

We assume the reader to be familiar with the basics of L systems (see, e.g., [9] and [14]) and L forms (see [18]). To establish the notations used throughout the paper we recall the central notions of the theory. First of all we recall the definition of an EOL form.

Definitions. An EOL *form* (*system*) F will be denoted as $F = (V, \Sigma, P, S)$ where V is its *total alphabet*, Σ is its *terminal alphabet*, $S \in V \setminus \Sigma$ is its *axiom*, and P is its set of *productions*.

F is called a *propagating* EOL *form*, abbreviated as EPOL *form*, if the right hand side of every production of F differs from the empty word, Λ.

F is called an $\{S, a\}$-EOL *form*, $\{S, a\}$*-form* for short, if $\#(V \setminus \Sigma) = \#\Sigma = 1$. As usual we write $\alpha \to x$ to denote an element of P and we write $\alpha \underset{P}{\to} x$ to abbreviate that $\alpha \to x$ belongs to P.

Furthermore, we use $\underset{F}{\Rightarrow}$ to denote the derivation relation induced by P; the symbols

$$\overset{k}{\underset{F}{\Rightarrow}}, \quad \overset{\leq k}{\underset{F}{\Rightarrow}}, \quad \overset{>k}{\underset{F}{\Rightarrow}} \ (k \text{ a positive integer}), \quad \overset{+}{\underset{F}{\Rightarrow}} \quad \text{and} \quad \overset{*}{\underset{F}{\Rightarrow}}$$

have the usual meaning. To avoid cumbersome notation we will often omit the specification F below the arrow.

F is called *synchronized* if for every $\alpha \in \Sigma$, $\alpha \overset{+}{\underset{F}{\Rightarrow}} x$ implies $x \notin \Sigma^*$.

The *language of* F is denoted by $L(F)$. □

EOL forms are used to define language families as follows.

Definitions. A substitution μ defined on some alphabet Δ is called a *dfl-substitution* (disjoint finite letter-substitution) if $\mu(\alpha)$ is a finite set of symbols for each $\alpha \in \Delta$ and $\mu(\alpha) \cap \mu(\beta) = \emptyset$ for $\alpha \neq \beta, \alpha, \beta \in \Delta$.

For a dfl-substitution μ and a set of productions P, we define

$$\mu(P) = \{y_1 \to y_2 : x_1 \underset{P}{\to} x_2, y_1 \in \mu(x_1) \text{ and } y_2 \in \mu(x_2)\}.$$

Let $F = (V, \Sigma, P, S)$ be an EOL form and μ a dfl-substitution on V. Further let $F' = (V', \Sigma', P', S')$ be an EOL system such that

(i) for every $\alpha \in V \setminus \Sigma$, $\mu(\alpha) \subseteq V' \setminus \Sigma'$,

(ii) for every $\alpha \in \Sigma$, $\mu(\alpha) \subseteq \Sigma'$,

(iii) $P' \subseteq \mu(P)$,
(iv) $S' \in \mu(S)$.
Then F' is called an *interpretation of F (modulo μ)*, abbreviated as $F' \lhd F(\mu)$. The *language family of F*, denoted $\mathscr{L}(F)$, is defined by $\mathscr{L}(F) = \{L(F') : F' \lhd F\}$. Two EOL forms F_1 and F_2 are called *form equivalent* if $\mathscr{L}(F_1) = \mathscr{L}(F_2)$. □

We mostly use the name 'form' whenever we discuss properties concerning language families, and the name 'system' whenever we discuss properties concerning languages.

In the paper we often deal with derivations.

Definitions. Let $F = (V, \Sigma, P, S)$ be an EOL system. A *derivation in F (starting from x_0)* is a sequence of words $(x_0, x_1, \ldots, x_n)$, $n \geq 1$ such that

$$x_0 \underset{F}{\Rightarrow} x_1, \quad x_1 \underset{F}{\Rightarrow} x_2, \ldots, x_{n-1} \underset{F}{\Rightarrow} x_n,$$

together with a precise description of how all occurrences in x_i are rewritten to obtain x_{i+1}, $0 \leq i \leq n-1$. Such a description can be formalized (see, e.g., [14]). We depict a derivation D by

$$D: \quad x_0 \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_n.$$

D is called *clean nonterminal* and we denote

$$D: \quad x_0 \overset{n}{\underset{\text{cnt}\, t}{\Rightarrow}} x_n$$

if for $0 < i < n$, $\operatorname{alph} x_i \cap (V \setminus \Sigma) \neq \emptyset$.

Each occurrence of a letter in every word from $\{x_0, \ldots, x_n\}$ has a unique contribution to x_n through D; if α is an occurrence of a letter in x_i, $0 \leq i \leq n$, then we use $\operatorname{ctr}_{D, x_i} \alpha$ to denote the contribution.

We also define $\operatorname{lhs} D = x_0$, $\operatorname{rhs} D = x_n$, $\operatorname{alph} D = \operatorname{alph} x_0 x_1 \cdots x_n$, $\operatorname{trace} D = \{x_0, x_1, \ldots, x_n\}$ and $\operatorname{itrace} D = \{x_1, \ldots, x_{n-1}\}$.

If p: $\alpha \underset{P}{\rightarrow} x$, then we also denote $\operatorname{lhs} p = \alpha$ and $\operatorname{rhs} p = x$. □

In the paper we often need 'isolations of derivations'. Given a number of derivations D_i (starting from a single symbol) we will construct derivations D_i' by renaming all symbols on the intermediate levels using new symbols (and each symbol will be used only once). Furthermore $\operatorname{lhs} D_i'$ ($\operatorname{rhs} D_i'$ respectively) results from $\operatorname{lhs} D_i$ ($\operatorname{rhs} D_i$ respectively) by a dfl-substitution μ. Formally we have the following.

Definitions. Let $F = (V, \Sigma, P, S)$ be an EOL form, $n \geq 1$ and for $1 \leq i \leq n$, let D_i be a derivation in F starting from α_i,

$$D_i: \quad \alpha_i \underset{F}{\Rightarrow} x_{i,1} \underset{F}{\Rightarrow} x_{i,2} \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{i,n_i-1} \underset{F}{\Rightarrow} x_{i,n_i}$$

such that $\alpha_i \in V$, $n_i \geq 1$ and $x_{i,n_i} = \beta_{i,1}\beta_{i,2}\cdots\beta_{i,m_i}$ where $m_i \geq 0$ and $\beta_{i,j} \in V$ for $1 \leq j \leq m_i$.

Let μ be a dfl-substitution on

$$V_1 = \{\alpha_i : 1 \leq i \leq n\} \cup \bigcup_{i=1}^{n} \operatorname{alph} x_{i,n_i}$$

and let Δ_n (Δ_t respectively) be an infinite enumerable alphabet of nonterminals (terminals respectively) such that $\Delta_n \cap \Delta_t = \emptyset$ and $(\Delta_n \cup \Delta_t) \cap \mu(V_1) = \emptyset$.

Then $(D_1', D_2', \ldots, D_n')$ is called an *isolated derivation tuple of* $(D_1, D_2, \ldots, D_n)$ *modulo* μ if the following conditions hold.

(i) For $1 \leq i \leq n$, D_i' results from D_i as follows. Represent D_i by its derivation tree. Replace now every node label β, except the labels of the root and the leaves, by an element of Δ_t (Δ_n respectively) if $\beta \in \Sigma$ ($\beta \in V \setminus \Sigma$ respectively). Replace the root label by $\alpha' \in \mu(\alpha)$ and for $1 \leq j \leq m_i$, replace the leaf label $\beta_{i,j}$ by $\beta_{i,j}' \in \mu(\beta_{i,j})$.

(ii) In the above construction no symbol of $\Delta_n \cup \Delta_t$ occurs twice (neither within one derivation tree nor in different derivation trees).

We denote

$$\operatorname{term}(D_1', D_2', \ldots, D_n') = \left(\bigcup_{i=1}^{n} \operatorname{alph} D_i\right) \cap (\Delta_t \cup \mu(\Sigma)),$$

$$\operatorname{nonterm}(D_1', D_2', \ldots, D_n') = \left(\bigcup_{i=1}^{n} \operatorname{alph} D_i\right) \cap (\Delta_n \cup \mu(V \setminus \Sigma)).$$

Let $F = (V, \Sigma, P, S)$ be an EOL form, let $n \geq 1$ and let $(D_1', D_2', \ldots, D_n')$ be an isolated derivation tuple of $(D_1, D_2, \ldots, D_n)$ modulo μ where $D_1, D_2, \ldots, D_n$ are derivations in F (starting from elements of V) such that the following conditions hold:

(a) $Q = \{\operatorname{lhs} D_i' \to \operatorname{rhs} D_i' : 1 \leq i \leq n\}$ constitutes a complete set of productions, i.e., for each $\alpha \in \bigcup_{i=1}^{n} \operatorname{alph} D_i'$, Q contains an α-production, and

(b) there exists an i, $1 \leq i \leq n$, such that $Z = \operatorname{lhs} D_i' \in \mu(S)$.

Then the EOL form $F' = (V', \Sigma', P', Z)$ is said to be *based on* $(D_1', D_2', \ldots, D_n')$ *and* Z if $V' = \operatorname{nonterm}(D_1', D_2', \ldots, D_n') \cup \operatorname{term}(D_1', D_2', \ldots, D_n')$, $\Sigma' = \operatorname{term}(D_1', D_2', \ldots, D_n')$, and P' equals the set of productions used in $D_1', D_2', \ldots, D_n'$.

Observe that if F' and F are as in the above definition, then $F' \lhd F(\mu)$. □

We now recall from [12] the notions of goodness and vompleteness, the central notions of the present paper.

Definitions. Let $F = (V, \Sigma, P, S)$ be an EOL form.

F is called *good* if whenever H is an EOL form such that $\mathscr{L}(H) \subseteq \mathscr{L}(F)$, then there exists an $F' \lhd F$ with $\mathscr{L}(H) = \mathscr{L}(F')$.

If F is not good, F is called *bad*.

F is called *complete* if $\mathscr{L}(F) = \mathscr{L}(\text{EOL})$.

F is called *vomplete* if F is good and complete. □

Recently we have proved that for an EOL form F it is decidable whether or not F is vomplete (see [17]). This was possible using both goodness and completeness of F. In trying to characterise good EOL forms much more difficulties arise. In the literature only few results are known concerning good but not vomplete EOL forms. E.g., synchronized EOL forms are bad (see, e.g., [12]) and 'most' propagating EOL forms are bad (see, e.g., [12]). Perhaps the most useful result in the area is the following lemma from [1].

Lemma 1.1. *Let* $F=(V,\Sigma,P,S)$ *be a good* EOL *form and let* $l\in \mathrm{LS}(L(F))$, $l\neq 0$. *Then there exists a word* $w\in L(F)$, $|w|=l$ *such that* $w \underset{F}{\overset{+}{\Rightarrow}} w$. $\square$

The above result only gives information concerning one length at the time. It can be generalized if we take two different lengths into account. This way we get the following theorem.

Theorem 1.1. *Let* $F=(V,\Sigma,P,S)$ *be a good* EOL *form and let* $k,l\in \mathrm{LS}(L(F))$, $k\neq l$, $k\neq 0$, $l\neq 0$. *Then there exists* $w_1, w_2\in L(F)$, $|w_1|=k$, $|w_2|=l$ *such that either* $w_1 \underset{F}{\overset{+}{\Rightarrow}} w_2$ *or* $w_2 \underset{F}{\overset{+}{\Rightarrow}} w_1$.

Proof. Let F, k and l be as in the statement of the theorem. Then Lemma 1.1 implies the existence of words $w_1, w_2\in L(F)$, $|w_1|=k$, $|w_2|=l$ such that

$$w_1 \underset{F}{\overset{+}{\Rightarrow}} w_1 \quad\text{and}\quad w_2 \underset{F}{\overset{+}{\Rightarrow}} w_2.$$

Consider now the following derivations in F.

$$\begin{aligned}
&D_1\colon\ S \overset{s_1}{\Rightarrow} w_1=\alpha_1\alpha_2\cdots\alpha_k, \qquad s_1\geq 1,\quad \alpha_i\in\Sigma \quad\text{for } 1\leq i\leq k;\\
&D_{w_1,1}\colon\ \alpha_1 \overset{t_1}{\Rightarrow} x_1,\ldots,D_{w_1,k}\colon\ \alpha_k \overset{t_1}{\Rightarrow} x_k, \qquad t_1\geq 1,\quad x_1x_2\cdots x_k=\alpha_1\alpha_2\cdots\alpha_k;\\
&D_2\colon\ S \overset{s_2}{\Rightarrow} w_2=\beta_1\beta_2\cdots\beta_l, \qquad s_2\geq 1,\ \beta_i\in\Sigma \quad\text{for } 1\leq k\leq l;\\
&D_{w_2,1}\colon\ \beta_1 \overset{t_2}{\Rightarrow} y_1,\ldots,D_{w_2,l}\colon\ \beta_l \overset{t_2}{\Rightarrow} y_l, \qquad t_2\geq 1,\ y_1y_2\cdots y_l=\beta_1\beta_2\cdots\beta_l.
\end{aligned}$$

Let

$$\Delta=\{a_1,a_2,\ldots,a_k,\ a_1',a_2',\ldots,a_k',\ b_1,b_2,\ldots,b_l\}.$$

Let μ be the dfl-substitution on V^* defined as follows.

(i) $\mu(S)=\{S\}$ and for $\alpha\in V\backslash(\Sigma\cup\{S\})$, $\mu(\alpha)=\emptyset$.

(ii) For $\alpha\in Z$,

$$\begin{aligned}\mu(\alpha)=\{a_i:\alpha_i=\alpha,\ 1\leq i\leq k\}\cup\{a_i':\alpha_i=\alpha,\ 1\leq i\leq k\}\\ \cup\{b_i:\beta_i=\alpha,\ 1\leq i\leq l\}.\end{aligned}$$

Let

$$\begin{aligned}
&E_1\colon\ S \overset{s_1}{\Rightarrow} a_1'a_2'\cdots a_k';\\
&E_{w_1,1}\colon\ a_1' \overset{t_1}{\Rightarrow} \bar{x}_1,\ldots,E_{w_1,k}\colon\ a_k' \overset{t_1}{\Rightarrow} \bar{x}_k;
\end{aligned}$$

$$E_2;\quad S \overset{s_2}{\Rightarrow} b_1 b_2 \cdots b_l;$$

$$E_{w_2,1}\colon\ b_1 \overset{t_2}{\Rightarrow} \bar{y}_1, \ldots, E_{w_2,l}\colon\ b_l \overset{t_2}{\Rightarrow} \bar{y}_l$$

be such that

$$(E_1, E_{w_1,1}, E_{w_1,2}, \ldots, E_{w_1,k}, E_2, E_{w_2,1}, E_{w_2,2}, \ldots, E_{w_2,l})$$

is an isolated derivation tuple of

$$(D_1, D_{w_1,1}, \ldots, D_{w_1,k}, D_2, D_{w_2,1}, \ldots, D_{w_2,l}) \quad \text{modulo } \mu$$

such that

$$\bar{x}_1 \bar{x}_2 \cdots \bar{x}_k = a_1 a_2 \cdots a_k \quad \text{and} \quad \bar{y}_1 \bar{y}_2 \cdots \bar{y}_l = b_1 b_2 \cdots b_l.$$

Furthermore for $1 \leq k \leq k$ let $E_{a_i}\colon a_i \Rightarrow z_i$ if

$$E_{w_1,i}\colon\quad a_1' \Rightarrow z_i \overset{t_1-1}{\Rightarrow} \bar{x}_i.$$

Finally let G be the EOL form based on

$$(E_1, E_{w_1,1}, E_{w_1,2}, \ldots, E_{w_1,k}, E_2, E_{w_2,1}, E_{w_2,2}, \ldots, E_{w_2,l}, E_{a_1}, E_{a_2}, \ldots, E_{a_k})$$

and S. Obviously $G \lhd F$. Let V_G denote the total alphabet of G. Without loss of generality assume that $k<l$.

Now let H be the EOL form which for all symbols of $V_G \setminus \{S, a_1, a_2, \ldots, a_k\}$ has the same productions as in G. The only S-production of H equals $S \to u$ where $S \Rightarrow u$ is the first derivation step of E_1. Furthermore, we have as productions $a_1 \to Z$ (Z a new symbol), $Z \to v$ where $S \Rightarrow v$ is the first derivation step of E_2 and $a_2 \to \Lambda, \ldots, a_k \to \Lambda$.

Obviously $L(H) = L(G)$ and comparing H and G one can easily prove that $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. Moreover if $K \in \mathscr{L}(H)$, then $\{k, l\} \subseteq \mathrm{LS}(K)$.

Since F is good, there exists an $F' \lhd F(\nu)$ such that $\mathscr{L}(F') = \mathscr{L}(H)$.

If for no pair (v_1, v_2) of words of $L(F')$ with $|v_1| = k$, $|v_2| = l$, $v_1 \overset{+}{\underset{F'}{\Rightarrow}} v_2$ or $v_2 \overset{+}{\underset{F'}{\Rightarrow}} v_1$, then an interpretation F'' of F' can be constructed, containing a word of length k and not a word of length l or containing a word of length l and not a word of length k. Since this is impossible, there must exist words $v_1, v_2 \in L(F')$ with $|v_1| = k$, $|v_2| = l$, $v_1 \overset{+}{\underset{F'}{\Rightarrow}} v_2$ or $v_2 \overset{+}{\underset{F'}{\Rightarrow}} v_1$. Then also

$$v^{-1}(v_1) \overset{+}{\underset{F}{\Rightarrow}} v^{-1}(v_2) \quad \text{or} \quad v^{-1}(v_2) \overset{+}{\underset{F}{\Rightarrow}} v^{-1}(v_1)$$

and hence the theorem holds. □

In the sequel we will often use the following lemma without an explicit reference to it.

Lemma 1.2. *Let $k \geq 2$ be an integer. Let $F = (V, \Sigma, P, S)$ and G be* EOL *forms such* *that $\mathscr{L}(F) \subseteq \mathscr{L}(G)$, $L(F) = \{w_1, \ldots, w_k\}$ and each language of $\mathscr{L}(G)$ contains at least k different words. Then there exists a derivation*

$$D: \quad S \overset{+}{\underset{F}{\Rightarrow}} w_{i_1} \overset{+}{\underset{F}{\Rightarrow}} w_{i_2} \overset{+}{\underset{F}{\Rightarrow}} \cdots \overset{+}{\underset{F}{\Rightarrow}} w_{i_k} \quad \textit{where } \{i_1, \ldots, i_k\} = \{1, \ldots, k\}.$$

Proof. Let F and G be as in the statement of the lemma. Then the lemma is proved by contraction. Assume that no derivation in F is like D of the statement of the lemma. Then let D_1: $S \overset{+}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} w$ be a derivation in F, $w \in L(F)$. Let μ be a dfl-substitution on V defined by $\mu(S) = \{S, Z\}$ and $\mu(\alpha) = \{\alpha\}$ if $\alpha \in V \setminus \{S\}$. Let $E: Z \overset{+}{\Rightarrow} w_1$ be such that (E) is an isolated derivation tuple of (D_1) modulo μ. Let $\bar{P}$ be a deterministic subset of P (i.e., for each letter $\alpha \in V$ fix an α-production). Finally let

$$F' = (V \cup \mathrm{nonterm}(E) \cup \mathrm{term}(E),\ \Sigma \cup \mathrm{term}(E),\ \bar{P} \cup \tilde{P}, Z)$$

where $\tilde{P}$ denotes the set of productions used in E. Then $L(F') \in \mathscr{L}(F) \subseteq \mathscr{L}(G)$ and $L(F')$ contains at most $(k-1)$ different words; a contradiction. Hence the lemma holds. □

For unexplained notions and notations we refer to [9] and [14] for L systems and [18] for L forms.

2. Terminal propagating forms

As a first class we will investigate terminal propagating forms. They are formally defined now.

Definition. Let $F = (V, \Sigma, P, S)$ be an EOL form.

F is called *terminal propagating* if for all $\alpha \in \Sigma$, $\alpha \overset{+}{\underset{F}{\Rightarrow}} x$ implies $x \neq \Lambda$.

If F is not terminal propagating, F is called *terminal erasing.* □

For an EOL system to be terminal propagating is a global property whereas the propagating restriction is a local property. Clearly using known decision results for EOL systems (see, e.g., [14]) for an EOL system F one can easily decide whether or not F is terminal propagating. In the case of $\{S, a\}$-forms the following lemma provides us with an easier decision procedure.

Lemma 2.1. *Let $F = (\{S, a\}, \{a\}, P, S)$ be an* EOL *form. Then F is terminal erasing if and only if one of the following conditions holds.*

(i) $a \underset{P}{\rightarrow} \Lambda$.

(ii) $\{a \rightarrow S^i, S \rightarrow \Lambda\} \subseteq P$ *for a positive integer i.*

Proof. Let F be as in the statement of the lemma. The 'if' part of the lemma is trivial. To prove the 'only if' part we proceed as follows. Since F is terminal erasing, $a \overset{+}{\underset{F}{\Rightarrow}} \Lambda$ must hold. If $a \to \Lambda$ is no production of P, clearly $S \underset{P}{\to} \Lambda$ and $a \underset{P}{\to} S^i$ for a positive integer i. $\square$

We will consider now terminal erasing $\{S, a\}$-forms. Such a form F can only be good if $a \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a$, as expressed by the following result.

Lemma 2.2. *Let $F = (\{S, a\}, \{a\}, P, S)$ be a good* EOL *form such that F is terminal propagating and $L(F) \neq \emptyset$.*
Then $a \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a$. Consequently $S \underset{P}{\to} a$ or $a \underset{P}{\to} a$.

Proof. Let F be as in the statement of the lemma. Clearly the first sentence implies the second sentence. We now prove the first sentence.

Since $L(F) \neq \emptyset$, $a^k \in L(F)$ for a positive integer k. Since a^k is the only word of length k in $L(F)$, Lemma 1.1 implies $a^k \overset{m}{\underset{F}{\Rightarrow}} a^k$ for a positive integer m. Then the fact that F is terminal propagating yields $a \overset{m}{\underset{F}{\Rightarrow}} a$. Let n_0 be the smallest positive integer n such that $a \overset{n}{\underset{F}{\Rightarrow}} a$. Thus D: $a \overset{n_0}{\underset{F}{\Rightarrow}} a$. We will now prove that D is clean nonterminal. If $n_0 = 1$, $a \overset{n_0}{\underset{\text{cnt}\, F}{\Rightarrow}} a$ clearly holds. Therefore assume that $n_0 > 1$. Then the fact that D is clean nonterminal is proved by contradiction.

For assume to the contrary that

$$D: \quad a \overset{i}{\underset{F}{\Rightarrow}} a^l \overset{n_0 - i}{\underset{F}{\Rightarrow}} a$$

for integers i, l where $1 \le i < n_0$. If $l < 1$ or $l > 1$, this contradicts the fact that F is terminal propagating. If $l = 1$ this contradicts the minimality of n_0. Thus $a \overset{n_0}{\underset{\text{cnt}\, F}{\Rightarrow}} a$. $\square$

Based on the above lemma we are now able to reduce drastically the number of terminal propagating $\{S, a\}$-forms which eventually can be good EOL forms. We have the following result.

Lemma 2.3. *Let $F = (\{S, a\}, \{a\}, P, S)$ be a good* EOL *form such that F is terminal propagating and $L(F) \neq \emptyset$. Then $L(F) = \{a\}$.*

Proof. (1) Let F be as in the statement of the lemma. Let k be a positive integer such that $S \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a^k$. (Such a k must exist otherwise $L(F) = \emptyset$.) We will prove that $k \in \{1, 2\}$. From Lemma 2.2 we get $a \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a$. Let

$$D_1: \quad S \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a^k, \qquad D_2: \quad a \overset{+}{\underset{\text{cnt}\, F}{\Rightarrow}} a$$

and let μ be the identity on $\{S, a\}^*$. Let E_1: $S \overset{+}{\Rightarrow} a^k$, E_2: $a \overset{+}{\Rightarrow} a$ be such that (E_1, E_2) is an isolated derivation tuple of (D_1, D_2) modulo μ. Finally let G be the EOL form based on (E_1, E_2) and S. Clearly $G \lhd F$ and $L(G) = \{a\} \in \mathscr{L}(F)$.

We will now prove by contradiction that $k \in \{1,2\}$. Assume that $k \geq 3$. Consider the EOL form

$$H:\quad S \to aba^{k-2},\quad a \to \Lambda,\quad b \to c^2d^{k-2},\quad c \to N,\quad d \to N,\quad N \to N.$$

Clearly $L(H) = \{aba^{k-2}, c^2d^{k-2}\}$ and $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. Then the goodness of F implies the existence of $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$.

Observe that each language of $\mathscr{L}(H)$ contains at least two words, one which is an interpretation of aba^{k-2} and another which is an interpretation of c^2d^{k-2}.

Hence either

$$aba^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} c^2d^{k-2} \quad \text{or} \quad c^2d^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} aba^{k-2}$$

for a positive integer t. Since F is terminal propagating, also F'' is. If

$$aba^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} c^2d^{k-2}, \quad \text{then} \quad a \underset{F''}{\overset{t}{\Rightarrow}} c, \quad b \underset{F''}{\overset{t}{\Rightarrow}} c$$

and thus $aba^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} c^k$, which contradicts the fact that $L(F'') = L(H)$. If

$$c^2d^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} aba^{k-2}, \quad \text{then} \quad c \underset{F''}{\overset{t}{\Rightarrow}} a, \quad d \underset{F''}{\overset{t}{\Rightarrow}} a$$

and thus $c^2d^{k-2} \underset{F''}{\overset{t}{\Rightarrow}} a^k$, which contradicts the fact that $L(F'') = L(H)$. Thus $k \in \{1,2\}$.

(2) As a next step we will prove by contradiction that $L(F)$ cannot contain a word of length greater than or equal to two.

Assume that $L(F)$ contains such a word. Then let $D\colon S \underset{F}{\overset{+}{\Rightarrow}} a^l$ be a derivation of shortest length such that $l \geq 2$. Then one of the following must hold.

$$D:\quad S \underset{\text{cnt}\,F}{\overset{t_1}{\Rightarrow}} a \underset{\text{cnt}\,F}{\overset{t_2}{\Rightarrow}} a^l, \quad l \geq 2, \tag{2.1}$$

or

$$E:\quad S \underset{\text{cnt}\,F}{\overset{t_1}{\Rightarrow}} a^2. \tag{2.2}$$

This is seen as follows. The proof of (1) yields the existence of $m \in \{1,2\}$ and positive integers t_1, t_4 such that

$$D:\quad S \underset{\text{cnt}\,F}{\overset{t_1}{\Rightarrow}} a^m \underset{F}{\overset{t_4}{\Rightarrow}} a^l.$$

If $m = 2$, then (2.2) holds. If $m = 1$, consider

$$\bar{D}:\quad a \underset{F}{\overset{t_4}{\Rightarrow}} a^l.$$

Clearly $a^n \in \operatorname{itrace} \bar{D}$, $n \geq 1$, cannot happen otherwise D can be shortened.

(2a) If (2.1) holds and $l > 2$, then consider the following EOL form.

$$H:\quad S \to a_1,\ a_1 \to aba^{l-2},\ a \to \Lambda,\ b \to c^2d^{l-2},\ c \to N,\ d \to N,\ N \to N.$$

Clearly $\mathscr{L}(H) \subseteq \mathscr{L}(F)$ and a contradiction can be derived as under (1).

(2b) If (2.2) holds, then let H be the following EOL form.

$$H:\quad S \to ab, \quad a \to \Lambda, \quad b \to ca, \quad c \to aa.$$

Obviously $L(H) = \{ab, ca, aa\}$. Using (2.2) and $a \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} a$ (see Lemma 2.2) we get $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. Moreover, each language of $\mathscr{L}(H)$ contains at least three different words.

Since F is good there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$. Let $F'' = (V'', \Sigma'', P'', S'')$. Since each language of $\mathscr{L}(H)$ must contain at least three different words we have

$$S'' \underset{F''}{\overset{+}{\Rightarrow}} u \underset{F''}{\overset{+}{\Rightarrow}} v \underset{F''}{\overset{+}{\Rightarrow}} w \quad \text{where } \{u, v, w\} = L(H).$$

(a) If $aa \underset{F''}{\overset{+}{\Rightarrow}} ab$ or $ca \underset{F''}{\overset{+}{\Rightarrow}} ab$, then $a \underset{F''}{\overset{+}{\Rightarrow}} b$ (recall that F and thus F'' is terminal propagating); Hence $aa \underset{F}{\overset{+}{\Rightarrow}} bb$ which contradicts $L(H) = L(F'')$.

(b) If $aa \underset{F''}{\overset{+}{\Rightarrow}} ca$ or $ab \underset{F''}{\overset{+}{\Rightarrow}} ca$, then $a \underset{F''}{\overset{+}{\Rightarrow}} c$. Hence $aa \underset{F''}{\overset{+}{\Rightarrow}} cc$ which contradicts $L(H) = L(F'')$.

From (a) and (b) it follows that $u \notin \{aa, ca, ab\}$ which causes the final contradiction. Hence (2.2) cannot hold.

(2c) If (2.1) holds and $l = 2$, then let H be the following EOL form.

$$H:\quad S \to d, \quad d \to ab, \quad a \to \Lambda, \quad b \to ca, \quad c \to aa.$$

Obviously $L(H) = \{d, ab, ca, aa\}$ and as in (2b) we can derive a contradiction.

From (2a), (2b) and (2c) it follows that $L(F) = \{a\}$ which proves the lemma. $\square$

The following lemma settles the only case which is left if we consider terminal propagating $\{S, a\}$-forms.

Lemma 2.4. *Let* $F = (\{S, a\}, \{a\}, P, S)$ *be a terminal propagating* EOL *form,* $L(F) = \{a\}$.

(i) *F is good if and only if* $a \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} a$.

(ii) $a \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} a$ *if and only if one of the following conditions holds.*

(ii.1) $\{S \to a, a \to S\} \subseteq P$.

(ii.2) $a \underset{P}{\to} a$.

Proof. Let F be as in the statement of the lemma.

(i) The 'only if' part immediately follows from Lemma 2.2. The 'if' part can be proved as in [12] using

$$S \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} a \quad \text{and} \quad a \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} a.$$

(ii) The 'if' part is trivial. The 'only if' part is proved as follows. Let $D\colon a \underset{\text{cnt}\,F}{\overset{t_0}{\Rightarrow}} a$ where t_0 is the minimal positive integer t such that $a \underset{\text{cnt}\,F}{\overset{t}{\Rightarrow}} a$. If $t_0 = 1$, $a \underset{P}{\to} a$. Thus assume $t_0 \neq 1$. Then $D\colon a \underset{F}{\Rightarrow} x \underset{F}{\overset{+}{\Rightarrow}} a$. Clearly $a \notin \text{alph}\, x$, otherwise t_0 could not be minimal. Consequently $x \in S^+$. If $x = S^k$, $k \geq 1$, then since we have $S \underset{F}{\overset{+}{\Rightarrow}} a$ also

$a^k \in L(F)$; a contradiction. Thus $x = S$ which implies $a \underset{P}{\rightarrow} S$. Hence the lemma holds. □

The following theorem summarizes the results of this section.

Theorem 2.1. *Let* $F = (\{S, a\}, \{a\}, P, S)$ *be a terminal propagating* EOL *form such that* $L(F) \neq \emptyset$. *Then it is decidable whether or not* F *is good.*

Proof. Let F be as in the statement of the theorem. Then to decide whether or not F is good, we use the following Diagram 1.

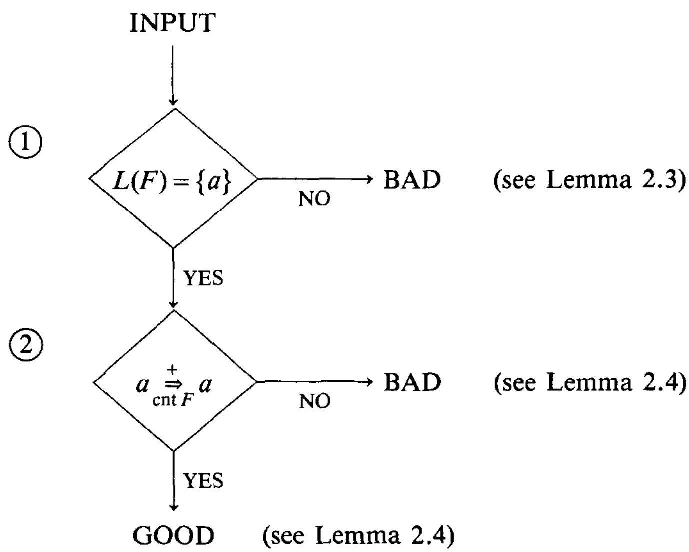

Diagram 1.

Clearly all decisions in Diagram 1 are effective (see, e.g., [14] for test 1 and [8] or Lemma 2.4 for test 2). Hence the theorem holds. □

3. Terminal erasing initial forcing free forms

In this section we start the investigation of the goodness of terminal erasing $\{S, a\}$-forms. First of all we divide the above class into two disjoint subclasses: initial forcing and initial forcing free forms. The latter forms are subject of the present section. Formally we have the following definition.

Definition. Let $F = (\{S, a\}, \{a\}, P, S)$ be an EOL form.

F is called *initial forcing* if $S \underset{P}{\rightarrow} x$ implies $s \in a^* \cup \{S\}$.

F is called *initial forcing free* if F is not initial forcing, i.e., if $S \underset{P}{\rightarrow} x$ where $\#_S x \geq 2$ or $\text{alph}\, x = \{S, a\}$. $\square$

For terminal erasing $\{S, a\}$-forms a result similar to Lemma 2.2 holds.

Lemma 3.1. *Let $F = (\{S, a\}, \{a\}, P, S)$ be a good* EOL *form such that F is terminal erasing and $L(F) \neq \emptyset$. Then $a \underset{\text{cnt}\, F}{\overset{+}{\Rightarrow}} a$. Consequently $S \underset{P}{\rightarrow} a$ or $a \underset{P}{\rightarrow} a$.*

Proof. Let F be as in the statement of the lemma. Clearly the first sentence implies the second sentence. We now prove the first sentence.

Let j be a positive integer such that D_1: $S \underset{\text{cnt}\, F}{\overset{+}{\Rightarrow}} a^j$. Lemma 2.1 implies that either $a \underset{P}{\rightarrow} \Lambda$, or $\{a \rightarrow S^i, S \rightarrow \Lambda\} \subseteq P$ for a positive integer i. If $a \underset{P}{\rightarrow} \Lambda$, then let D_2: $a \underset{F}{\Rightarrow} \Lambda$; if $\{a \rightarrow S^i, S \rightarrow \Lambda\} \subseteq P$ for a positive integer, then let D_2: $a \underset{F}{\Rightarrow} S^i \underset{F}{\Rightarrow} \Lambda$.

Let μ be the identity on $\{S, a\}^*$ and let E_1: $S \overset{+}{\Rightarrow} a^j$, E_2: $a \overset{+}{\Rightarrow} \Lambda$ be such that (E_1, E_2) is an isolated derivation tuple of (D_1, D_2) modulo μ. Finally let G be the EOL form based on (E_1, E_2) and S. Obviously $G \triangleleft F$ and $L(G) = \{a^j\} \in \mathscr{L}(F)$. Consider the EOL form

$$H: \quad S \rightarrow a^j, \quad a \rightarrow b, \quad b \rightarrow N, \quad N \rightarrow N.$$

Clearly $L(H) = \{a^j, b^j\}$ and $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. Then the goodness of F implies the existence of $F'' \triangleleft F' \triangleleft F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$.

Observe that each language of $\mathscr{L}(H)$ contains at least two words, one which is an interpretation of a^j and another which is an interpretation of b^j. Hence either $a^j \underset{F''}{\overset{t}{\Rightarrow}} b^j$ or $b^j \underset{F''}{\overset{t}{\Rightarrow}} a^j$ for a positive integer t. Without loss of generality assume that $a^j \underset{F''}{\overset{t}{\Rightarrow}} b^j$. Then $a \underset{F''}{\overset{t}{\Rightarrow}} b$ must hold, otherwise $L(F'')$ would contain words of exceeding j. Thus, using inverse interpretation, $a \underset{F}{\overset{t}{\Rightarrow}} a$. Let n_0 be the smallest positive integer such that $a \underset{F}{\overset{n}{\Rightarrow}} a$. Thus we have D: $a \underset{F}{\overset{n_0}{\Rightarrow}} a$. D must be clean nonterminal otherwise n_0 would not be minimal. Hence $a \underset{\text{cnt}\, F}{\overset{+}{\Rightarrow}} a$ holds. $\square$

In the study of terminal erasing initial forcing free $\{S, a\}$-forms it will be very useful to consider such vomplete forms separately. To this aim we recall the following result from [17].

Theorem 3.1. (i) *$F = (V, \Sigma, P, S)$ be an* EOL *form, let k be a positive integer and let $b \in \Sigma$ such that the following conditions hold.*

(i.1) $\quad S \underset{\text{cnt}\, F}{\overset{+}{\Rightarrow}} x_1 b x_2, \quad x_1 x_2 \in V^*(V \setminus \Sigma)V^*,$

(i.2) $\quad b \underset{\text{cnt}\, F}{\overset{k}{\Rightarrow}} b,$

(i.3) $\quad b \underset{\text{cnt}\, F}{\overset{k}{\Rightarrow}} x_3 b x_4 b x_5, \quad x_3 x_4 x_5 \in V^*(V \setminus \Sigma)V^*,$

(i.4) $b \underset{\text{cnt}\,F}{\overset{>k}{\Rightarrow}} \Lambda$, *and*

(i.5) $bx_1x_2x_3x_4x_5 \overset{\leq k}{\Rightarrow} \Lambda$.

Then F is vomplete.

(ii) *Let $F=(V,\Sigma,P,S)$ be a vomplete* EOL *form. Then for any pair of integers (p,q) where $p\geq 3$ and $q\geq 2$, there is a $b\in\Sigma$ and a positive integer k such that the following conditions hold.*

(ii.1) $S \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} b^p$,

(ii.2) $b \underset{\text{cnt}\,F}{\overset{k}{\Rightarrow}} b$,

(ii.3) $b \underset{\text{cnt}\,F}{\overset{k}{\Rightarrow}} b^q$, *and*

(ii.4) $b \underset{\text{cnt}\,F}{\overset{>k}{\Rightarrow}} \Lambda$. □

As a corollary of the above we get for $\{S,a\}$-forms the following result.

Corollary 3.1. *Let $F=(\{S,a\},\{a\},P,S)$ be an* EOL *form. Then F is vomplete if and only if the following five conditions hold.*

(C.1) $S \underset{P}{\rightarrow} \Lambda$,

(C.2) $a \underset{P}{\rightarrow} \Lambda$ *or* $a \underset{P}{\rightarrow} S^i$, $i\geq 1$,

(C.3) $S \underset{P}{\rightarrow} w_1Sw_2$, $w_1w_2\in\{S,a\}^+$,

(C.4) $a \underset{P}{\rightarrow} w_3Sw_4$, $w_3w_4\in\{S,a\}^*$,

(C.5) $S \underset{\text{cnt}\,F}{\overset{\geq 2}{\Rightarrow}} a$.

Proof. (1) We first prove that (C.1)–(C.5) imply (i.1)–(i.5) from Theorem 3.1 and hence yield the vompleteness of F. Condition (C.3) yields

$$S \underset{P}{\rightarrow} u_1au_2Su_3 \text{ or } S \underset{P}{\rightarrow} u_1Su_2au_3 \text{ or } S \underset{P}{\rightarrow} u_1Su_2Su_3,\quad u_1,u_2,u_3\in\{S,a\}^*.$$

From (C.5) we get $S \underset{\text{cnt}\,F}{\overset{m}{\Rightarrow}} a$ for an integer $m\geq 2$. Let $k=m+3$. Then there are two cases to consider

(a) $S \underset{P}{\rightarrow} u_1au_2Su_3$ (the case $S \underset{P}{\rightarrow} u_1Su_2au_3$ is symmetric). Then clearly (i.1) holds. Moreover, we also have $a \underset{\text{cnt}\,F}{\overset{2}{\Rightarrow}} uSv \underset{F}{\Rightarrow} \bar{u}u_1au_2Su_3\bar{v} \underset{\text{cnt}\,F}{\overset{m}{\Rightarrow}} a$,

$$a \underset{F}{\Rightarrow} w_3Sw_4 \underset{F}{\Rightarrow} \bar{w}_3u_1au_2Su_3\bar{w}_4 \underset{F}{\Rightarrow} \bar{u}_1w_3Sw_4\bar{u}_2u_1au_2Su_3\bar{u}_3 \underset{\text{cnt}\,F}{\overset{m}{\Rightarrow}} au'Sv'a,$$

$$a \underset{\text{cnt}\,F}{\overset{m+3}{\Rightarrow}} u''Sv'' \underset{\text{cnt}\,F}{\overset{+}{\Rightarrow}} \Lambda, \quad \text{and} \quad aS \overset{\leq 2}{\Rightarrow} \Lambda$$

where $u, v, u', v', u'', v'' \in \{S, a\}^*$. Thus (i.1) through (i.5) hold.

(b) $S \underset{P}{\Rightarrow} u_1 S u_2 S u_3$. Then

$$S \underset{F}{\Rightarrow} u_1 S u_2 S u_3 \overset{m}{\underset{\text{cnt}\,F}{\Rightarrow}} a u''' S v''',$$

$$a \overset{2}{\underset{\text{cnt}\,F}{\Rightarrow}} u S v \underset{F}{\Rightarrow} \bar{u} u_1 S u_2 S u_3 \bar{v} \overset{m}{\underset{\text{cnt}\,F}{\Rightarrow}} a,$$

$$a \underset{F}{\Rightarrow} w_3 S w_4 \underset{F}{\Rightarrow} \bar{w}_3 u_1 S u_2 S u_3 \bar{w}_4 \underset{F}{\Rightarrow} \bar{u}_1 u_1 S u_2 S u_3 \bar{u}_2 u_1 S u_2 S u_3 \bar{u}_3 \overset{m}{\underset{\text{cnt}\,F}{\Rightarrow}} a a u''' S v''',$$

$$a \overset{m+3}{\underset{\text{cnt}\,F}{\Rightarrow}} u'' S v'' \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} \Lambda, \quad \text{and} \quad aS \overset{\leq 2}{\underset{F}{\Rightarrow}} \Lambda$$

where $u, v, u', v', u'', v'', u''', v''' \in \{S, a\}^*$. Thus (i.1) through (i.5) hold.

(2) Next we prove that (C.1)–(C.5) are necessary conditions for the vompleteness of F. Consider (ii) of Theorem 3.1. Let $p, q > \max\{\text{maxr}\, F, 3\}$. One can easily see that: the choice of p and (ii.1) gives (C.3), the choice of q and (ii.3) gives $k \geq 2$ and (C.4), and (ii.4) gives (C.1) and (C.2).

To prove (C.5) we proceed as follows. From the above it follows that either

$$S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} xSyaz \underset{F}{\Rightarrow} a^p \quad \text{or} \quad S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} xaySz \underset{F}{\Rightarrow} a^p \quad \text{or} \quad S \overset{+}{\underset{F}{\Rightarrow}} xSySz \underset{F}{\Rightarrow} a^p$$

where $x, y, z \in \{S, a\}^*$ and the distinguished occurrences of S and a have nonempty contributions to a^p. Thus either

$$S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} xSyaz \overset{k}{\underset{\text{cnt}\,F}{\Rightarrow}} a,$$

or

$$S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} xaySz \overset{k}{\underset{\text{cnt}\,F}{\Rightarrow}} a,$$

or

$$S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} xSySz \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} \tilde{u} a^p \tilde{v} w_1 S w_2 \tilde{w} \overset{k}{\underset{\text{cnt}\,F}{\Rightarrow}} a.$$

(We have used the fact that $Sa \overset{\leq 2}{\underset{F}{\Rightarrow}} \Lambda$, (ii.2) and (C.3)). Hence (C.5) holds. $\square$

In the following rather technical lemma we prove that 'quite a number' of good EOL forms must have erasing productions for nonterminals. It is a generalization of the main theorem of [4] where it is proved that vomplete EOL forms must have the above property.

Lemma 3.2. *Let p be a positive integer and let H be the* EOL *form defined by the following productions.*

$$\begin{aligned} H:\quad & S \to a^{2p} b_1 b_2 \cdots b_p, \quad S \to e^p b_1 b_2 \cdots b_p, \quad a \to X, \\ & b_1 \to d_1 d_2 \cdots d_p, \quad b_2 \to \Lambda, \ldots, b_p \to \Lambda, \\ & d_1 \to c^p, \quad d_1 \to \Lambda, \ldots, d_p \to \Lambda, \\ & c \to c^{5p}, \quad e \to e^{3p}, \quad X \to \Lambda. \end{aligned}$$

Each EOL *form which is form equivalent with H must contain a production* $A \to \Lambda$ *where* A *is a nonterminal.*

Proof. Let p and H be as in the statement of the lemma. Observe that

$$L(H) = \{a^{2p}b_1b_2\cdots b_p, e^pb_1b_2\cdots b_p, e^{p(3p)}d_1d_2\cdots d_p\}$$
$$\cup\{c^{p(5p)^n} : n\geq 0\} \cup \{e^{p(3p)^n} : n\geq 2\} \cup \{e^{p(3p)^{n+2}}c^{p(5p)^n} : n\geq 0\}.$$

Also observe that for every length in LS($L(H)$) there exists only one word in $L(H)$ of this length and for each word w of $L(H)$ there is only one derivation of w in H.

The lemma is now proved by contradiction using analogous arguments as in the main theorem of [4]. We proceed as follows. Assume that there exists an EOL form $G=(V, \Sigma, P, S)$ such that $\mathscr{L}(G)=\mathscr{L}(F)$ and for each $A\in V\backslash\Sigma$, $A\to\Lambda$ does not belong to P. Then there exists a $G'=(V', \Sigma', P', S')\lhd G$ such that $L(G')=L(H)$.

(1) *If* $S \underset{H}{\overset{+}{\Rightarrow}} x \underset{H}{\overset{+}{\Rightarrow}} y$, *$x$ and y terminal words, then either*

$$S' \underset{G'}{\overset{+}{\Rightarrow}} x \underset{G'}{\overset{+}{\Rightarrow}} y \quad \textit{or} \quad S' \underset{G'}{\overset{+}{\Rightarrow}} y \underset{G'}{\overset{+}{\Rightarrow}} x.$$

Proof of (1). See, e.g., [4].

(2) $$a^{2p}b_1b_2\cdots b_p \underset{G'}{\overset{+}{\Rightarrow}} c^p.$$

Proof of (2). $S \underset{H}{\Rightarrow} a^{2p}b_1b_2\cdots b_p \underset{H}{\overset{+}{\Rightarrow}} c^p$. By (1), either $a^{2p}b_1b_2\cdots b_p \underset{G'}{\overset{+}{\Rightarrow}} c^p$ or $c^p \underset{G'}{\overset{+}{\Rightarrow}} a^{2p}b_1b_2\cdots b_p$. The second alternative is clearly impossible.

(3) $$a \underset{G'}{\overset{n}{\Rightarrow}} \Lambda \quad \textit{and} \quad b_1b_2\cdots b_p \underset{G'}{\overset{n}{\Rightarrow}} c^p \quad \textit{for some } n\geq 1.$$

Proof of (3). By (2) we have

$$D:\ a^{2p}b_1b_2\cdots b_p \underset{G'}{\overset{n}{\Rightarrow}} c^p, \quad n\geq 1.$$

If there are occurrences $a^{(1)}$ and $a^{(2)}$ of a such that

$$\mathrm{ctr}_{D, a^{2p}b_1b_2\cdots b_p}\, a^{(1)} = c^{i_1}, \qquad \mathrm{ctr}_{D, a^{2p}b_1b_2\cdots b_p}\, a^{(2)} = c^{i_2},$$

and $i_1<i_2$, then

$$a^{2p}b_1b_2\cdots b_p \underset{G'}{\overset{n}{\Rightarrow}} c^{p+i_2-i_1}.$$

The latter clearly cannot be the case since $p<p+i_2-i_1\leq 2p$. Thus all occurrences of a must have the same contribution to c^p. Hence (3) easily follows.

(4) $$c \underset{G'}{\overset{*}{\Rightarrow}} \Lambda,\ e \underset{G'}{\overset{*}{\Rightarrow}} \Lambda \quad \textit{are impossible.}$$

Proof of (4). See, e.g., [4].

(5) $$e^{p(3p)}d_1d_2\cdots d_p \underset{G'}{\overset{*}{\Rightarrow}} e^{p(3p)^2} \quad \textit{and} \quad e^pb_1b_2\cdots b_p \underset{G'}{\overset{*}{\Rightarrow}} e^{p(3p)}d_1d_2\cdots d_p.$$

Proof of (5). We have $e^{p(3p)}d_1d_2\cdots d_p \overset{*}{\underset{H}{\Rightarrow}} e^{p(3p)^2}$. Since $e^{p(3p)^2} \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)}d_1d_2\cdots d_p$ is impossible by (4), we have

$$e^{p(3p)}d_1d_2\cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)^2} \qquad \text{by (1).}$$

We have

$$e^pb_1b_2\cdots b_p \overset{*}{\underset{H}{\Rightarrow}} e^{p(3p)}d_1d_2\cdots d_p.$$

Since $e^{p(3p)}d_1d_2\cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} d^pb_1b_2\cdots b_p$ is impossible by (4), we have

$$e^pb_1b_2\cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3)}d_1d_2\cdots d_p \qquad \text{by (1).}$$

(6) $\qquad e \overset{*}{\underset{G'}{\Rightarrow}} e^k, \quad 1 \le k < 3p$ *is impossible.*

Proof of (6). For the proof that $e \overset{*}{\underset{G'}{\Rightarrow}} e$ is impossible, see, e.g., [4]. If $e \overset{*}{\underset{G'}{\Rightarrow}} e^k$, $2 \le k < 3p$, then $e^{p(3p)^2} \overset{*}{\underset{G'}{\Rightarrow}} e^{kp(3p)^2}$. Since $p(3p)^2 < kp(3p)^2 < p(3p)^3$ this contradicts the fact that $L(H) = L(G')$. Thus (6) holds.

(7) $\qquad d_1d_2\cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} \Lambda.$

Proof of (7). From (5) we get

$$D\colon\quad e^{p(3p)}d_1d_2\cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)^2}.$$

If in D there is a d_i, $1 \le i \le p$ such that $\mathrm{ctr}_{D, e^{p(3p)}d_1d_2\cdots d_p} d_i \ne \Lambda$, then there is an occurrence $e^{(1)}$ of e such that $\mathrm{ctr}_{D, e^{p(3p)}d_1d_2\cdots d_p} e^{(1)} = e^j$ for some $0 \le j < 3p$, contradicting (4) or (6).

(8) $\{c \to \Lambda, e \to \Lambda\} \cap P' = \emptyset$ *and it is not true that* $\{b_1 \to \Lambda, b_2 \to \Lambda, \ldots, b_p \to \Lambda\} \subseteq P'$.

Proof of (8). $c \to \Lambda$ and $e \to \Lambda$ are not in P' by (4). By (5) we have

$$D\colon\quad e^pb_1b_2\cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)}d_1d_2\cdots d_p.$$

Obviously $\mathrm{ctr}_{D, e^pb_1b_2\cdots b_p} e \in e^*$. Moreover if $e^{(1)}$ and $e^{(2)}$ are two different occurrences of e in $e^pb_1b_2\cdots b_p$, then

$$\mathrm{ctr}_{D, e^pb_1b_2\cdots b_p} e^{(1)} = \mathrm{ctr}_{D, e^pb_1b_2\cdots b_p} e^{(2)}.$$

Then from (6) we get $\mathrm{ctr}_{D, e^pb_1b_2\cdots b_p} e = e^{3p}$ for each occurrence of e in $e^pb_1b_2\cdots b_p$.

If $\{b_1 \to \Lambda, b_2 \to \Lambda, \ldots, b_p \to \Lambda\} \subseteq P'$, then $e^pb_1b_2\cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)}$; a contradiction.

(9) $\qquad c^p \overset{*}{\underset{G'}{\Rightarrow}} d_1d_2\cdots d_p$ *does not hold.*

Proof of (9). Clear since $c^p \in L(H)$ but $d_1 d_2 \cdots d_p \notin L(H)$

(10) *Let* D: $S' \underset{G'}{\Rightarrow} x_1 \underset{G'}{\Rightarrow} x_2 \underset{G'}{\Rightarrow} \cdots \underset{G'}{\Rightarrow} x_n$, $n \geq 1$ *and* $x_i = e^p b_1 b_2 \cdots b_p$ *for some* $1 \leq i \leq n$, *then* $x_j = e^{p(3p)} d_1 d_2 \cdots d_p$ *for some* $1 \leq j \leq n$ *or* $x_n \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)} d_1 d_2 \cdots d_p$.

Proof of (10). See, e.g., [4].

(11) *For every derivation*

$$D: \quad b_1 b_2 \cdots b_p \underset{G'}{\Rightarrow} x_1 \underset{G'}{\Rightarrow} x_2 \underset{G'}{\Rightarrow} \cdots \underset{G'}{\Rightarrow} x_{n-1} \underset{G'}{\Rightarrow} x_n = c^p, \quad n \geq 1,$$

there exists an $1 \leq i \leq n-1$ *with* $x_i = d_1 d_2 \cdots d_p$.

Proof of (11). Consider a derivation D as above ((3) yields the existence of such a derivation). Since $e^p b_1 b_2 \cdots b_p \in L(G')$ we can use D to obtain D' as follows.

$$D': \quad S' \overset{*}{\underset{G'}{\Rightarrow}} e^p b_1 b_2 \cdots b_p \underset{G'}{\Rightarrow} u_1 x_1 \underset{G'}{\Rightarrow} \cdots \underset{G'}{\Rightarrow} u_{n-1} x_{n-1} \underset{G'}{\Rightarrow} u_n c^p \underset{G'}{\Rightarrow} \cdots.$$

$e^{p(3p)} d_1 d_2 \cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} e^p b_1 b_2 \cdots b_p$ is impossible by (4); $c \overset{*}{\underset{G'}{\Rightarrow}} \Lambda$ does not hold by (4); $c^p \overset{*}{\underset{G'}{\Rightarrow}} d_1 d_2 \cdots d_p$ is impossible by (9). Clearly $c \overset{*}{\underset{G'}{\Rightarrow}} x$, $\operatorname{alph} x \subseteq \{e, d_1, d_2, \ldots, d_p\}$ and $\#\operatorname{alph} x \geq 2$ is also impossible. From the above observations and (10) it follows that

$$e^p b_1 b_2 \cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} e^{p(3p)} d_1 d_2 \cdots d_p \overset{*}{\underset{G'}{\Rightarrow}} u_n c^p.$$

As in (8) we get that, if

$$E: \quad e^p b_1 b_2 \cdots b_p \overset{*}{\underset{G}{\Rightarrow}} e^{p(3p)} d_1 d_2 \cdots d_p$$

we get $\operatorname{ctr}_{E, e^p b_1 b_2 \cdots b_p} e = e^{3p}$ for each occurrence of e. Thus $b_1 b_2 \cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} d_1 d_2 \cdots d_p$ has been used in D', hence in D, concluding the argument.

(12) $\qquad S' \overset{*}{\underset{G'}{\Rightarrow}} uXv d_1 d_2 \cdots d_p$ *and* $uXv \overset{*}{\underset{G'}{\Rightarrow}} \Lambda$ *for* u, $v \in V'^*$, $X \in V' \setminus \Sigma'$.

Proof of (12). By (2)

$$S' \overset{*}{\underset{G'}{\Rightarrow}} a^{2p} b_1 b_2 \cdots b_p \overset{n}{\underset{G'}{\Rightarrow}} c^p, \quad n \geq 1.$$

Moreover we have

$$D_1: \ a \overset{n}{\underset{G'}{\Rightarrow}} \Lambda \quad \text{and} \quad D_2: \ b_1 b_2 \cdots b_p \underset{G'}{\Rightarrow} c^p \qquad \text{by (3).}$$

By (11) we have

$$D_2: \ b_1 b_2 \cdots b_p \overset{n_1}{\underset{G'}{\Rightarrow}} d_1 d_2 \cdots d_p \overset{n_2}{\underset{G'}{\Rightarrow}} c^p \quad \text{for } n_1 + n_2 = n.$$

Thus

$$S' \overset{*}{\underset{G'}{\Rightarrow}} a^{2p} b_1 b_2 \cdots b_p \overset{n_1}{\underset{G'}{\Rightarrow}} w^{2p} d_1 d_2 \cdots d_p \overset{n_2}{\underset{G'}{\Rightarrow}} c^p.$$

Since $e^{p(3p)} d_1 d_2 \cdots d_p$ is the only word of $L(G')$ ending on $d_1 d_2 \cdots d_p$ and $e^{p(3p)} = w^{2p}$

is impossible (otherwise $e \overset{n_2}{\underset{G'}{\Rightarrow}} \Lambda$ which contradicts (4)), w^{2p} contains a nonterminal X. Thus (12) holds.

(13) *$a \to \Lambda$ does not occur in P'.*

Proof of (13). Since $b_1 b_2 \cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} d_1 d_2 \cdots d_p$ by (11), $a \underset{P'}{\to} \Lambda$ would allow to derive

$$a^{2p} b_1 b_2 \cdots b_p \overset{*}{\underset{G'}{\Rightarrow}} d_1 d_2 \cdots d_p \notin L(G').$$

(14) *It is not true that $\{d_1 \to \Lambda, d_2 \to \Lambda, \ldots, d_p \to \Lambda\} \subseteq P'$.*

Proof of (14). Assume that $\{d_1 \to \Lambda, d_2 \to \Lambda, \ldots, d_p \to \Lambda\} \subseteq P'$. By (12) and (7) we have

$$S' \overset{*}{\underset{G'}{\Rightarrow}} uXvd_1 d_2 \cdots d_p \underset{G'}{\Rightarrow} x \overset{*}{\underset{G'}{\Rightarrow}} y \underset{G'}{\Rightarrow} \Lambda, \quad X \in V' \setminus \Sigma'.$$

We have $x \neq \Lambda$, since $X \to \Lambda$ does not belong to P', y is chosen such that $y \neq \Lambda$. But then $y \in (\Sigma' \setminus \{a, c, e\})^+ \cap L(G')$ by (8), (13) and since $X \to \Lambda$ does not belong to P'. Thus $y \in \{b_1, b_2, \ldots, b_p, d_1, d_2, \ldots, d_p\}^+ \cap L(G')$. Since no such y is in $L(G')$ we have a contradiction.

We are now finally in a position to combine observations (1) through (14) into a contradiction. Consider the derivations D_1: $uXv \overset{n}{\underset{G'}{\Rightarrow}} \Lambda$ ($n \geq 2$ and n minimal), D_2: $S' \overset{*}{\underset{G'}{\Rightarrow}} uXvd_1 d_2 \cdots d_p$ as in (12), and D_3: $d_1 d_2 \cdots d_p \overset{m}{\Rightarrow} \Lambda$ ($m \geq 2$ and m minimal) by (7) and (14). On the basis of D_1 and D_3 we have

$$uXv \overset{n-1}{\underset{G'}{\Rightarrow}} x_1 \underset{G'}{\Rightarrow} \Lambda \quad \text{and} \quad d_1 d_2 \cdots d_p \overset{m-1}{\underset{G'}{\Rightarrow}} x_2 \underset{G'}{\Rightarrow} \Lambda$$

where $x_1, x_2 \in \{\Sigma' \setminus \{a, c, e\})^+$ as in the proof of (14). Then using D_2 followed by D_1 and D_3 we obtain

$$S' \overset{*}{\underset{G'}{\Rightarrow}} uXvd_1 d_2 \cdots d_p \overset{\max(n-1, m-1)}{\underset{G'}{\Rightarrow}} x_3$$

where x_3 is equal to x_1, x_2 or $x_1 x_2$. Since $\{x_1, x_2, x_1 x_2\} \cap L(G') = \emptyset$ this is a contradiction which completes the proof of the lemma. □

Using the previous results we are able to show that for a large class of terminal erasing initial forcing free $\{S, a\}$-forms the containment of the production $S \to \Lambda$ separates bad from vomplete forms. Formally we have the following two results.

Lemma 3.3. *Let $F = (\{S, a\}, \{a\}, P, S)$ be a terminal erasing initial forcing free* EOL *form, $L(F) \neq \emptyset$ such that the following conditions hold.*

(i) $a \underset{P}{\to} y$, $S \in \text{alph}\, y$.

(ii) *$S \to \Lambda$ does not belong to P.*

Then F is bad.

Proof. Let F be as in the statement of the lemma. Since $L(F) \neq \emptyset$ and $S \to \Lambda$ is not a production of P, it must be the case that $S \underset{P}{\rightarrow} a^j$, $j \geq 1$. From Lemma 2.1 we get $a \underset{P}{\rightarrow} \Lambda$. Let $S \underset{P}{\rightarrow} x$ be as in the definition of initial forcing free form (i.e., $\#_S x \geq 2$ or $\operatorname{alph} x = \{S, a\}$) and let $a \underset{P}{\rightarrow} y$ be as in the statement of the lemma. Let h be the homomorphism on $\{S,a\}^*$ defined by $h(S)=x$ and $h(a)=y$. For each positive integer n, and $1 \leq l < \#_S h^n(S)$ define

$$D^n_{S,l}:\quad S \underset{\text{cnt}\,F}{\overset{n}{\Rightarrow}} h^n(S) = u_0 S u_1 S u_2 \cdots S u_{k_1} \underset{F}{\Rightarrow} x^l (a^j)^{k_1 - l} \underset{F}{\Rightarrow} a^{jql}$$

where $k_1 = \#_S h^n(S)$, $u_i \in a^*$ for $0 \leq i \leq k_1$ and $q = \#_S x$.

Also for each positive integer n, and $1 \leq l < \#_S h^n(a)$ define

$$D^n_{a,l}:\quad \mathrm{a} \underset{\text{cnt}\,F}{\overset{n}{\Rightarrow}} h^n(a) = v_0 S v_1 S v_2 \cdots S v_{k_2} \underset{F}{\Rightarrow} x^l (a^j)^{k_2 - l} \underset{F}{\Rightarrow} a^{jql}$$

where $k_2 = \#_S h^n(a)$, $v_i \in a^*$ for $0 \leq i \leq k_2$ and $q = \#_S x$.

Observe that $D^n_{S,l}$ and $D^n_{a,l}$ as defined above are clean nonterminal derivations. Let $p = jq$. Let

$$D_{S,\text{block}}:\ S \underset{F}{\Rightarrow} x, \qquad D_{a,\text{block}}:\ a \underset{F}{\Rightarrow} y \quad \text{and} \quad D_{a,\Lambda}:\ a \underset{F}{\Rightarrow} \Lambda.$$

Then define the dfl-substitutions μ and μ_{block} on $\{S,a\}^*$ by

$$\mu(S) = \{S, N\}, \qquad \mu(a) = \{a, b_1, b_2, \ldots, b_p, c, d_1, d_2, \ldots, d_p, e, f\},$$
$$\mu_{\text{block}}(S) = \{N\} \quad \text{and} \quad \mu_{\text{block}}(a) = \{f\}.$$

Let

$$\begin{array}{ll}
E_1:\ S \overset{+}{\Rightarrow} a^p a^p b_1 b_2 \cdots b_p, & E_2:\ S \overset{+}{\Rightarrow} e^p b_1 b_2 \cdots b_p,\\
E_3:\ S \overset{+}{\Rightarrow} c^p, & E_4:\ b_1 \overset{+}{\Rightarrow} d_1 d_2 \cdots d_p,\\
E_5:\ c \overset{+}{\Rightarrow} c^{5p}, & E_6:\ d_1 \overset{+}{\Rightarrow} c^p,\\
E_7:\ e \overset{+}{\Rightarrow} e^{3p}, & E_8:\ a \Rightarrow \mu_{\text{block}}(y),\\
E_9:\ N \Rightarrow \mu_{\text{block}}(x), & E_{10}:\ f \Rightarrow \mu_{\text{block}}(y),\\
E_{11,k}:\ b_k \Rightarrow \Lambda \quad \text{for } 2 \leq k \leq p, & \\
E_{12,k}:\ d_k \Rightarrow \Lambda \quad \text{for } 1 \leq k \leq p, &
\end{array}$$

be such that $(E_1, E_2, \ldots, E_{10}, E_{11,2}, \ldots, E_{11,p}, E_{12,2}, \ldots, E_{12,p})$ is an isolated derivation tuple of

$$\tau = (D^n_{S,3}, D^n_{S,2}, D^n_{S,1}, D^n_{a,1}, D^n_{a,5}, D^n_{a,1}, D^n_{a,3}, D_{a,\text{block}}, D_{S,\text{block}}, D_{a,\text{block}}, D_{a,\Lambda}, \ldots, D_{a,\Lambda})$$

modulo μ where n is such that $\#_S h^n(S) > 6$ and $\#_a h^n(S) > 6$.

Finally let G be the EOL form based on τ and S. Clearly $G \lhd F$. Let H be the EOL form from the statement of Lemma 3.2. Then comparing G and H one can easily conclude that $\mathscr{L}(H) \subseteq \mathscr{L}(F)$.

The fact that F is bad is now proved by contradiction. Assume that F is good. Then the fact that $\mathscr{L}(H) \subseteq \mathscr{L}(F)$ implies the existence of an EOL form $F' \lhd F$ such that $\mathscr{L}(H)=\mathscr{L}(F')$. Since F and hence F' contains no production $A \to \Lambda$ where A is a nonterminal, this contradicts Lemma 3.2. Thus F must be bad. □

Lemma 3.4. *Let $F=(\{S,a\},\{a\},P,S)$ be a terminal erasing initial forcing free* EOL *form, $L(F)\neq\emptyset$ such that the following conditions hold.*

(i) $a \underset{P}{\rightarrow} y$, $S\in \operatorname{alph} y$.

(ii) $S \underset{P}{\rightarrow} \Lambda$.

Then F is good if and only if F is vomplete.

Proof. The 'if' part is trivial. To prove the 'only if' part we proceed as follows. Let F be a good EOL form which satisfies the assumptions of the lemma. Then clearly (C.1), (C.3) and (C.4) from the statement of Corollary 3.1 are satisfied; (C.2) holds by Lemma 3.1. Moreover Lemma 2.1 yields $a \underset{P}{\rightarrow} \Lambda$ or $a \underset{P}{\rightarrow} S^i$, $i\geq 1$, Lemma 3.1 implies $S \underset{P}{\rightarrow} a$ or $a \underset{P}{\rightarrow} a$, and $L(F)\neq\emptyset$ implies $S \overset{t}{\underset{\text{cnt}\,F}{\Rightarrow}} a^k$ for some positive integers t and k.

For all possible cases one can easily verify that (C.5) also holds. Hence F is vomplete. □

In the statements of both Lemma 3.3 and Lemma 3.4 the condition $a \underset{P}{\rightarrow} y$, $S\in \operatorname{alph} y$ occurs. The following lemma gives conditions which force such production to be included in the set of productions.

Lemma 3.5. *Let $F=(\{S,a\},\{a\},P,S)$ be a good terminal erasing initial forcing free* EOL *form, $L(F)\neq\emptyset$. Then $a \underset{P}{\rightarrow} y$, $S\in \operatorname{alph} y$.*

Proof. From Lemma 3.1 we get $a \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} a$. The lemma is now proved by contradiction. Assume that $a \underset{P}{\rightarrow} y$ implies $S\notin \operatorname{alph} y$. Thus

$$a \overset{*}{\underset{F}{\Rightarrow}} y \quad \text{implies} \quad y\in a^*.$$

Then we get $a \underset{P}{\rightarrow} a$. Let k be the smallest positive integer k' such that $S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} a^{k'}$. Thus

$$D_1:\quad S \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} s^k. \tag{3.1}$$

D_2 is defined as follows. In case (i), i.e., $S \underset{P}{\rightarrow} x$, $\#_S x=m\geq 2$, let h be the homomorphism on $\{S,a\}^*$ defined by $h(S)=x$ and $h(a)=a$. Let n_0 be the smallest positive integer n such that $\#_S h^n(S)>\operatorname{maxr} F$. Let $q=\#_S h^{n_0}(S)$. Then let

$$D_2:\quad S \overset{n_0}{\underset{\text{cnt}\,F}{\Rightarrow}} h^{n_0}(S) \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} a^{qk+r}=a^p \quad \text{where } r=\#_a h^{n_0}(S).$$

In case (ii), i.e., $S \underset{P}{\rightarrow} x$, $\#_S x=1$, $\#_a x=m\geq 1$, let h be the homomorphism on $\{S,a\}^*$ defined by $h(S)=x$ and $h(a)=a$. Let n_0 be the smallest positive integer n such that $\#_a h^n(S)>k(\operatorname{maxr} F)$. Let $q=\#_a h^{n_0}(S)$. Then let

$$D_2:\quad S \overset{n_0}{\underset{\text{cnt}\,F}{\Rightarrow}} h^{n_0}(S) \overset{+}{\underset{\text{cnt}\,F}{\Rightarrow}} a^{q+k}=a^p.$$

Let D_3: $a \Rightarrow a$. In all cases let μ be the identity on $\{S,a\}^*$ and let E_1: $S \overset{+}{\Rightarrow} a^k$, E_2: $S \overset{+}{\Rightarrow} a^p$, E_3: $a \Rightarrow a$ be such that (E_1, E_2, E_3) is an isolated derivation tuple of (D_1, D_2, D_3). Finally let G be the EOL form based on (E_1, E_2, E_3) and S. Clearly $G \lhd F$ and $L(G) = \{a^k, a^p\} \in \mathscr{L}(F)$.

Consider the EOL form

$$H:\quad S \to a_1 a_2 \cdots a_k, \quad a_1 \to a^p, \quad a_j \to \Lambda \quad \text{for } 2 \le j \le k, \quad a \to N, \quad N \to N.$$

Clearly $L(H) = \{a_1 a_2 \cdots a_k,\ a^p\}$ and $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. Then the goodness of F implies the existence of $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$.

Observe that the language of $\mathscr{L}(H)$ contains at least two words, one which is an interpretation of $a_1 a_2 \cdots a_k$ and another which is an interpretation of a^p. Hence

$$\text{either} \quad a_1 a_2 \cdots a_k \underset{F''}{\overset{+}{\Rightarrow}} a^p \quad \text{or} \quad a^p \underset{F''}{\overset{+}{\Rightarrow}} a_1 a_2 \cdots a_k.$$

The latter is clearly impossible; the former can also not be the case since $p > k(\operatorname{maxr} F)$ and (3.1) yield the existence of a terminal word w such that $k < |w| < p$; a contradiction. Hence the lemma holds. □

We are now ready to state the main result of the section.

Theorem 3.2. *Let* $F = (\{S,a\}, \{a\}, P, S)$ *be a terminal erasing initial forcing free* EOL *form such that* $L(F) \neq \emptyset$. *Then it is decidable whether or not* F *is good.*

Proof. Let F be as in the statement of the theorem. Then to decide whether or not F is good, we use the following Diagram 2. Tests 1 and 2 are obviously effective; test 5 is effective by [17] or Corollary 3.1. Hence the theorem holds. □

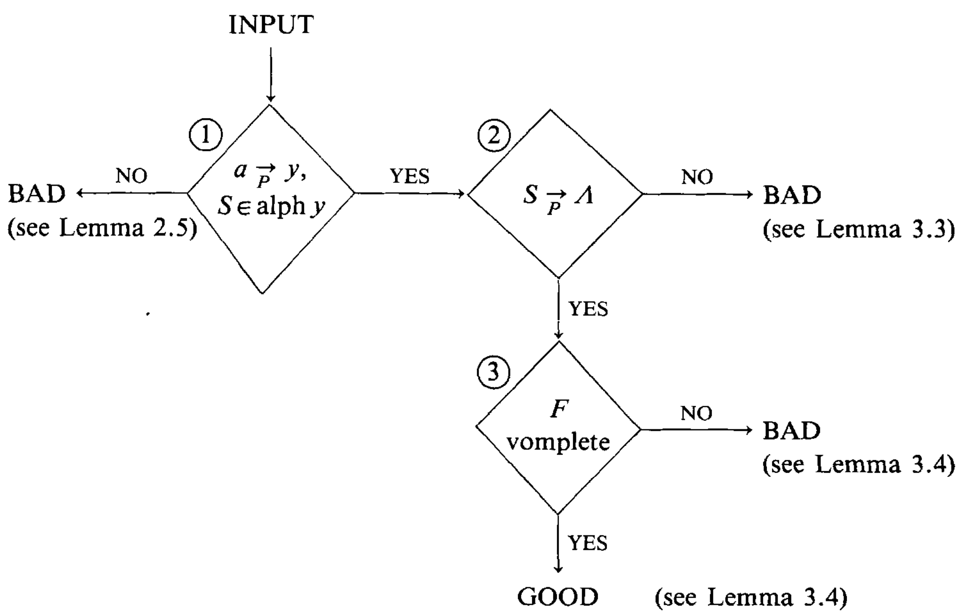

Diagram 2.

4. Terminal erasing initial forcing forms

In this section we will investigate terminal erasing initial forcing $\{S,a\}$-forms. As a first result we have the following lemma.

Lemma 4.1. *Let* $F=(\{S,a\},\{a\},P,S)$ *be a terminal erasing initial forcing* EOL *form,* $L(F)\neq\emptyset$ *such that the following conditions hold.*

(i.1) $a \underset{P}{\to} x$, $\operatorname{alph} x=\{S,a\}$, *or*

(i.2) $a \underset{P}{\to} x$, $\#_S x\geq 2$ *and* $S \underset{P}{\to} S$.

(ii) $S\to\Lambda$ *does not belong to* P.

Then F *is bad.*

Proof. Let F be as in the statement of the lemma. Since F is terminal erasing, (ii) and Lemma 2.1 yield $a \underset{P}{\to} \Lambda$. Since $L(F)\neq\emptyset$ and F is initial forcing, $S \underset{P}{\to} a^k$ for some $k\geq 1$. Let $j=\max\{k : S \underset{P}{\to} a^k\}$.

If (i.1) holds, then let h be the homomorphism on $\{S,a\}^*$ defined by $h(S)=a^j$ and $h(a)=x$. For each positive integer n, and $1\leq l\leq \#_a h^n(a)$ define

$$D^n_{a,l}:\ a \underset{\operatorname{cnt} F}{\overset{n}{\Rightarrow}} h^n(a)=u_0au_1au_2\cdots au_{k_1} \underset{F}{\Rightarrow} h(u_0au_1a\cdots au_l)h(u_{l+1}u_{l+2}\cdots u_{k_1}) \underset{F}{\Rightarrow} a^{jql}$$

where $k_1=\#_a h^n(a)$, $u_i\in S^*$ for $0\leq i\leq k_1$ and $q=\#_S x$.

If (i.2) holds and (i.1) is not the case, then $a \underset{P}{\to} S^i$, $i\geq 2$. Then for each positive integer n define

$$D^n_a:\quad a \underset{F}{\Rightarrow} x_1=S^i \underset{F}{\Rightarrow} x_2=a^jS^{i-1} \underset{F}{\Rightarrow} x_3=S^{ij}S^{i-1} \underset{F}{\Rightarrow} x_4=a^jS^{ij-1}S^{i-1} \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2n}=a^jS^t \underset{F}{\Rightarrow} x_{2n+1}=S^{ij+t}$$

Furthermore for $1\leq l\leq \#_S x_{2n+1}=l'$ define

$$D^n_{a,l}:\quad a \underset{\operatorname{cnt} F}{\overset{2n+1}{\Rightarrow}} x_{2n+1}=S^{l'} \underset{F}{\Rightarrow} (a^j)^{l'-l}S^l \underset{F}{\Rightarrow} a^{jl}.$$

Also observe that we have D_Λ: $a \underset{F}{\Rightarrow} \Lambda$, $D_{a,\text{block}}$: $a \underset{F}{\Rightarrow} x$, and if (i.2) holds $D_{S,\text{block}}$: $S \underset{F}{\Rightarrow} S$.

Using isolated derivations of the above derivations one can easily prove (as in the proof of Lemma 3.3) that $\mathscr{L}(\bar{H})\subseteq\mathscr{L}(F)$ where $\bar{H}$ is the EOL form which is defined as follows.

$$\begin{aligned}\bar{H}:\quad & S\to a_1a_2\cdots a_j,\quad a_1\to a^{2p}b_1b_2\cdots b_p,\quad a_1\to e^pb_1b_2\cdots b_p,\\ & a_2\to\Lambda,\ldots,a_j\to\Lambda,\quad a\to X,\quad b_1\to d_1d_2\cdots d_p,\quad b_2\to\Lambda,\ldots,b_p\to\Lambda,\\ & d_1\to c^p,\quad d_1\to\Lambda,\quad d_2\to\Lambda,\ldots,d_p\to\Lambda,\quad c\to c^{5p},\quad e\to e^{3p},\quad X\to\Lambda\end{aligned}$$

with $p=2jq$.

As in Lemmas 3.2 and 3.3 we will prove that F is bad. This is done by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(\bar{H})$ and $L(F'') = L(\bar{H})$. Let H be the EOL form from the proof of Lemma 3.2 with $p = 2jq$.

(1) $L(\bar{H}) = L(H) \cup \{a_1 a_2 \cdots a_j\}$ and $L(H)$ contains no word of length i.

(2) Points (1) through (13) from the proof of Lemma 3.2 remain valid if we replace G' by $F'' = (V'', \Sigma'', P'', S'')$.

(3) Also point (14) remains true since $S'' \overset{*}{\underset{F''}{\Rightarrow}} y \underset{F''}{\Rightarrow} \Lambda$ with y a nonempty terminal word would imply

$$y \in \{b_1, b_2, \ldots, b_p, d_1, d_2, \ldots, d_p, a_1, \ldots, a_j\}^* \cap L(F'').$$

This is only possible if $y = a_1 a_2 \cdots a_j$. But then the choice of j, p and the fact that F (and thus F'') is initial forcing yields

$$S'' \overset{+}{\underset{\mathrm{cnt}\, F''}{\Rightarrow}} a_1 a_2 \cdots a_j \underset{F''}{\Rightarrow} \Lambda.$$

Consequently $\{a_1 a_2 \cdots a_j\} \in \mathscr{L}(F'') \subseteq \mathscr{L}(\bar{H})$; a contradiction. Hence (14) holds.

(4) The final contradiction is then obtained following the final argument of Lemma 3.2 combined with a reasoning as under (3). Consequently F must be bad. $\square$

The above lemma is now generalized to the case where $S \to \Lambda$ belongs to P.

Lemma 4.2. *Let* $F = (\{S, a\}, \{a\}, P, S)$ *be a terminal erasing initial forcing* EOL *form,* $L(F) \neq \emptyset$ *such that one of the following conditions holds.*

(i) $a \underset{P}{\to} x$, $\operatorname{alph} x = \{S, a\}$.

(ii) $a \underset{P}{\to} x$, $\#_S x \geq 2$, $S \underset{P}{\to} S$ *and* (i) *does not hold.*

Then F *is bad.*

Proof. (1) Let F be as in the statement of the lemma. If $S \to \Lambda$ does not belong to P, then by Lemma 4.1, F is bad. Therefore without loss of generality assume that $S \underset{P}{\to} \Lambda$. Also, because of Lemma 3.1 we can assume $S \underset{P}{\to} a$ or $a \underset{P}{\to} a$. Since $L(F) \neq \emptyset$, $\{S \to a^k : k \geq 1\} \cap P \neq \emptyset$.

Let $j = \max\{k : k \geq 1$ and $S \underset{P}{\to} a^k\}$. Moreover, since F is terminal erasing, we have $a \underset{P}{\to} \Lambda$ or $a \underset{P}{\to} S^i$ for some positive integer i. If (i) is the case, let then $a \underset{P}{\to} x$ be as in (i) of the statement of the lemma. We have the following cases to consider.

(i.1) $\{S \to \Lambda,\ S \to a^j,\ a \to x,\ a \to \Lambda,\ S \to a\} \subseteq P$.

(i.2) $\{S \to \Lambda,\ S \to a^j,\ a \to x,\ a \to \Lambda,\ a \to a\} \subseteq P$.

(i.3) $\{S \to \Lambda,\ S \to a^j,\ a \to x,\ a \to S^i,\ S \to a\} \subseteq P$.

(i.4) $\{S \to \Lambda,\ S \to a^j,\ a \to x,\ a \to S^i,\ a \to a\} \subseteq P$.

If (ii) is the case, let then $a \to S^i$, $i \geq 2$. We have the following cases to consider.

(ii.1) $\{S \to \Lambda,\ S \to a^j,\ a \to S^i,\ S \to S,\ S \to a\} \subseteq P$.

(ii.2) $\{S \to \Lambda,\ S \to a^j,\ a \to S^i,\ S \to S,\ a \to a\} \subseteq P.$

In all cases we will prove the existence of an integer $n \geq 2$ such that

$$a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j+1}, \qquad a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j} \quad \text{and} \quad a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j-1}.$$

To see this we first prove that for $n_1 \geq 2$ and $n_2 < n_1$, $a^{n_1} \overset{2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{n_2}$. We consider cases (i.1) through (ii.2) separately.

(i.1) and (i.2): $a^{n_1} \underset{F}{\Rightarrow} x^{n_1} \underset{F}{\Rightarrow} a^{n_2}$.

(i.3) and (ii.1): $a^{n_1} \underset{F}{\Rightarrow} S^{in_1} \underset{F}{\Rightarrow} a^{n_2}$.

(i.4) and (ii.2): $a^{n_1} \underset{F}{\Rightarrow} a^{n_2}S^{(n_1-n_2)i} \underset{F}{\Rightarrow} a^{n_2}$.

In cases (i.1) through (i.4) let h be the homomorphism on $\{S, a\}^*$ defined by $h(a) = x$ (x as in (i) of the statement of the lemma) and $h(S) = a^j$. Let m be a positive integer such that $\#_a h^m(a) > 2j+1$. Then for $p \in \{2j-1, 2j, 2j+1\}$ define

$$D_{a,p}\colon \quad a \overset{m}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} h^m(a) = u_0au_1au_2 \cdots au_{k_1} \overset{2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^p$$

where $k_1 = \#_a h^m(a)$ and for $0 \leq k \leq k_1$, $u_k \in S^*$.

In cases (ii.1) and (ii.2) define

$$D_a\colon \quad a \underset{F}{\Rightarrow} x_1 = S^i \underset{F}{\Rightarrow} x_2 = a^jS^{i-1} \underset{F}{\Rightarrow} x_3 = S^{ij}S^{i-1} \underset{F}{\Rightarrow} x_4 = a^jS^{ij-1}S^{i-1} \underset{F}{\Rightarrow} \cdots.$$

Let m be a positive integer such that $\#_S x_{2m+1} > 2j+2$. Then for $p \in \{2j-1, 2j, 2j+1\}$ define

$$D_{a,p}\colon \quad a \overset{2m+1}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} x_{2m+1} = S^t \underset{F}{\Rightarrow} Sa^{j(t-1)} \overset{2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^p.$$

Thus in all cases there is an integer $n \geq 2$ such that we have

$$D_{a,2j-1}\colon\ a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j-1}, \qquad D_{a,2j}\colon\ a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j} \quad \text{and} \quad D_{a,2j+1}\colon\ a \overset{n}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} a^{2j+1}.$$

Furthermore we have

$$D_{a,\Lambda}\colon \quad a \overset{\leq 2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} \Lambda \quad \text{and} \quad D_S\colon \quad S \underset{F}{\Rightarrow} a^j.$$

(2) We first consider the case where $j > 1$. Then let μ be the dfl-substitution on $\{S, a\}^*$ defined by $\mu(S) = \{S\}$ and $\mu(a) = \{a, b, c\}$. Let

$$E_1\colon \quad S \to ab^{j-1}, \qquad E_2\colon \quad a \overset{n}{\Rightarrow} ac^{2j}, \qquad E_3\colon \quad a \overset{n}{\Rightarrow} ab^{j-1}c^{j-1},$$

$$E_4\colon \quad b \overset{+}{\Rightarrow} \Lambda, \qquad E_5\colon \quad c \overset{+}{\Rightarrow} \Lambda$$

be such that $(E_1, E_2, E_3, E_4, E_5)$ is an isolated derivation tuple of $(D_S, D_{a,2j+1}, D_{a,2j-1}, D_{a,\Lambda}, E_{a,\Lambda})$ modulo μ and let G be the EOL form based on $(E_1, E_2, E_3, E_4, E_5)$ and

S. Clearly $G \lhd F$ and $L(G) = \{ab^{j-1}, ac^{2j}, ab^{j-1}c^{j-1}\}$. Let H be the following EOL form.

$$H\colon \quad S \to ac^{2j}, \quad a \to ab^{j-1}, \quad b \to c, \quad c \to \Lambda.$$

One can easily see that $L(H) = L(G)$ and comparing the productions of H and G yields $\mathscr{L}(H) \subseteq \mathscr{L}(G) \subseteq \mathscr{L}(F)$. The fact that F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$. Let $F'' = (V'', \Sigma'', P'', S'')$. Observe that

(4.1) Each language of $\mathscr{L}(H)$ contains at least three words which are interpretations of ac^{2j}, ab^{j-1} and $ab^{j-1}c^{j-1}$ respectively.

The fact that F is initial forcing together with the choice of j yields

$$D\colon \quad S'' \underset{\operatorname{cnt} F}{\overset{+}{\Rightarrow}} ab^{j-1}.$$

Let Z be a new symbol. Let ν be the dfl-substitution on V''^* defined by $\nu(S'') = \{S'', Z\}$ and $\nu(\alpha) = \{\alpha\}$ for $\alpha \in V'' \setminus \{S''\}$. Let $E\colon Z \overset{+}{\Rightarrow} ab^{j-1}$ be such that (E) is an isolated derivation tuple of D modulo ν. Let P''' be a deterministic complete subset of P'' (i.e., for each $\alpha \in V''$, P''' contains exactly one α''-production of P''). Then define the EOL form

$$F''' = (V'' \cup \operatorname{term}(E) \cup \operatorname{nonterm}(E), \Sigma'' \cup \operatorname{term}(E), P''' \cup \tilde{P}, Z)$$

where $\tilde{P}$ denotes the set of productions used in E.

Clearly $F''' \lhd F'' \lhd F'$, thus $\mathscr{L}(F''') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $\{ab^{j-1}\} \subseteq L(F''') \subseteq L(F'') = L(H)$. Then (4.1) yields $L(F''') = L(H)$. Then since F''' is deterministic and since $\mathscr{L}(F''') \subseteq \mathscr{L}(H)$ there must be positive integers t_1 and t_2 such that either

(a) $$Z \underset{\operatorname{cnt} F'''}{\overset{+}{\Rightarrow}} ab^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_1}{\Rightarrow}} ac^{2j} \underset{\operatorname{cnt} F'''}{\overset{t_2}{\Rightarrow}} ab^{j-1}c^{j-1}, \quad \text{or}$$

(b) $$Z \underset{\operatorname{cnt} F'''}{\overset{+}{\Rightarrow}} ab^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_1}{\Rightarrow}} ab^{j-1}c^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_2}{\Rightarrow}} ac^{2j}.$$

If (1) holds, then clearly

$$c \underset{F'''}{\overset{t_2}{\Rightarrow}} \Lambda \quad \text{and} \quad a \underset{F'''}{\overset{t_2}{\Rightarrow}} ab^{j-1}c^{j-1}.$$

Then

$$ab^{j-1} \underset{F'''}{\overset{t_2}{\Rightarrow}} ab^{j-1}c^{j-1}y_1^{j-1} \underset{F'''}{\overset{t_1}{\Rightarrow}} ac^{2j}y_2^{j-1}y_3^{j-1} \neq ab^{j-1}c^{j-1}$$

which contradicts the fact that F''' is deterministic.

If (b) holds then consider

$$\bar{E}\colon \quad Z \underset{\operatorname{cnt} F'''}{\overset{+}{\Rightarrow}} ab^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_1}{\Rightarrow}} ab^{j-1}c^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_1}{\Rightarrow}} ab^{j-1}y^{j-1} \underset{\operatorname{cnt} F'''}{\overset{t_1}{\Rightarrow}} \cdots.$$

Then the word ac^{2j} does not occur in trace $\bar{E}$ which again contradicts the determinism of F'''. Hence F is bad in the case $j > 1$.

(3) Next we consider the case $j=1$. Then as above let μ be the dfl-substitution on $\{S, a\}^*$ defined by $\mu(S)=\{S\}$ and $\mu(a)=\{a, b, c\}$. Let

$$E_1\colon\ S \Rightarrow a, \qquad E_2\colon\ a \overset{n}{\Rightarrow} acb, \qquad E_3\colon\ a \overset{n}{\Rightarrow} ba, \qquad E_4\colon\ b \overset{+}{\Rightarrow} \Lambda, \qquad E_5\colon\ c \overset{+}{\Rightarrow} \Lambda$$

be such that $(E_1, E_2, E_3, E_4, E_5)$ is an isolated derivation tuple of $(D_S, D_{a,2j+1}, D_{a,2j}, D_{a,\Lambda}, D_{a,\Lambda})$ modulo μ and let G be the EOL form based on $(E_1, E_2, E_3, E_4, E_5)$ and S. Clearly $G \lhd F$ and $L(G)=\{a, acb, ba\}$. Let H be the following form.

$$H\colon\quad S \to acb, \quad c \to b, \quad b \to a, \quad a \to \Lambda.$$

One can easily see that $L(H)=L(G)$ and comparing the productions of H and G yields $\mathscr{L}(H) \subseteq \mathscr{L}(G) \subseteq \mathscr{L}(F)$. The fact that F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F''') \subseteq \mathscr{L}(F')=\mathscr{L}(H)$ and $L(F'')=L(H)$. Let $F''=(V'', \Sigma'', P'', S'')$.

Observe that each language of $\mathscr{L}(H)$ contains at least three words which are interpretations of a, acb and ba respectively. As in the case $j>1$ we define F''' with axiom Z, F''' deterministic, $L(F''')=L(H)$ and $\mathscr{L}(F''') \subseteq \mathscr{L}(H)$. As above we have either

(a) $$Z \underset{\mathrm{cnt}\,F'''}{\overset{+}{\Rightarrow}} a \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} acb \underset{\mathrm{cnt}\,F'''}{\overset{t_2}{\Rightarrow}} ba \quad \text{or}$$

(b) $$Z \underset{\mathrm{cnt}\,F'''}{\overset{+}{\Rightarrow}} a \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} ba \underset{\mathrm{cnt}\,F'''}{\overset{t_2}{\Rightarrow}} acb.$$

But then, if (a) holds,

$$Z \underset{\mathrm{cnt}\,F'''}{\overset{+}{\Rightarrow}} a \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} acb \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} acby_1y_2 \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} \cdots,$$

a contradiction, and if (b) holds,

$$Z \underset{\mathrm{cnt}\,F'''}{\overset{+}{\Rightarrow}} a \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} ba \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} y_1ba \underset{\mathrm{cnt}\,F'''}{\overset{t_1}{\Rightarrow}} \cdots,$$

a contradiction. Hence F is bad in the case $j=1$. This ends the proof of the lemma. $\square$

Next we investigate what happens if we consider terminal erasing initial forcing EOL forms where the conditions of Lemma 4.2 are not satisfied. We have to consider EOL forms $F=(\{S, a\}, \{a\}, P, S)$ such that

$$P \subseteq \{S \to S\} \cup \{S \to a^k : k \geq 0\} \cup \{a \to S^k : k \geq 0\} \cup \{a \to a^k : k \geq 0\}.$$

Moreover we can assume that if $a \underset{P}{\to} x$ and $\#_S x \geq 2$ then $S \to S$ does not belong to P.

In the rest of the section we will use the following notations.

$$\begin{aligned} j &= \max\{k : S \underset{P}{\to} a^k\} && \text{if it exists otherwise } j \text{ is undefined,}\\ i &= \max\{k : a \underset{P}{\to} S^k\} && \text{if it exists otherwise } i \text{ is undefined,}\\ l &= \max\{k : a \underset{P}{\to} a^k\} && \text{if it exists otherwise } l \text{ is undefined.}\end{aligned}$$

In the following lemma we investigate the case $ijl>1$.

Lemma 4.3. *Let $F=(\{S,a\},\{a\},P,S)$ be a terminal erasing initial forcing* EOL *form such that $L(F)\neq\emptyset$ and $a \underset{P}{\rightarrow} x$ implies $x\in a^*\cup S^*$. Moreover, if $i\geq 2$, then $S\to S$ does not belong to P. If $ijl>1$, then F is bad.*

Proof. Let F, i, j, l be as in the statement of the Lemma and let $ijl>1$.

(1) $j>1$. Let $D_{a,1}$ and $D_{a,2}$ be the following infinite derivations.

$$D_{a,1}:\quad a \underset{F}{\Rightarrow} x_1 = S^i \underset{F}{\Rightarrow} x_2 = a^{ij} \underset{F}{\Rightarrow} x_3 = S^{i^2j} \underset{F}{\Rightarrow} x_4 = a^{i^2j^2} \underset{F}{\Rightarrow} \cdots.$$

$$D_{a,2}:\quad a \underset{F}{\Rightarrow} y_1 = a^l \underset{F}{\Rightarrow} y_2 = S^{li} \underset{F}{\Rightarrow} y_3 = a^{lij} \underset{F}{\Rightarrow} y_4 = S^{li^2j} \underset{F}{\Rightarrow} \cdots.$$

Let $m\geq 2$ be an integer such that $\#_a x_{2m} - \max\{i^2j^3, i^2j^2l\}$. Then, if $S \underset{P}{\rightarrow} \Lambda$, let

$$D_{a,1,1}:\quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m} = a^{i^mj^m} \underset{F}{\Rightarrow} x_{2m+1} = S^{i^{m+1}j^m} \underset{F}{\Rightarrow} a^{i^2j^2};$$

$$D_{a,1,2}:\quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m+1} \underset{F}{\Rightarrow} a^{i^{m+1}j^{m+1}},$$

$$D_{a,2,1}:\quad a \underset{F}{\Rightarrow} y_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} y_{2m} = S^{li^mj^{m-1}} \underset{F}{\Rightarrow} \Lambda.$$

If $a \underset{P}{\rightarrow} \Lambda$, let

$$D_{a,1,1}:\quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m} = a^{i^mj^m} \underset{F}{\Rightarrow} S^{i^2j} \underset{F}{\Rightarrow} a^{i^2j^2};$$

$$D_{a,1,2}:\quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m} \underset{F}{\Rightarrow} x_{2m+1} \underset{F}{\Rightarrow} a^{i^{m+1}j^{m+1}};$$

$$D_{a,2,1}:\quad a \underset{F}{\Rightarrow} y_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} y_{2m} \underset{F}{\Rightarrow} y_{2m+1} \underset{F}{\Rightarrow} \Lambda.$$

Furthermore let

$$D_S:\quad S \underset{F}{\Rightarrow} a^j \quad\text{and}\quad D_{a,\Lambda}:\quad a \overset{\leq 2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} \Lambda.$$

Let μ be the dfl-substitution on $\{S,a\}^*$ defined by $\mu(S)=\{S\}$ and $\mu(a)=\{a,b,c,d\}$. Let $p=i^{m+1}j^{m+1}$,

$$E_1:\quad S\Rightarrow ab^{j-1},\qquad E_2:\quad a\overset{+}{\Rightarrow} c^{i^2j^2},\qquad E_3:\quad a\overset{+}{\Rightarrow} d^p,$$
$$E_4:\quad b\overset{+}{\Rightarrow}\Lambda,\qquad E_5:\quad c\overset{+}{\Rightarrow}\Lambda,\qquad E_6:\quad d\overset{+}{\Rightarrow}\Lambda$$

be such that $(E_1,E_2,E_3,E_4,E_5,E_6)$ is an isolated derivation tuple of $\tau=(D_S,D_{a,1,1}, D_{a,1,2},D_{a,2,1},D_{a,\Lambda},D_{a,\Lambda})$ modulo μ; let G be the EOL form based on τ and S. Clearly $G\lhd F$ and a close inspection of E_1 through E_6 yields $L(G)=\{ab^{j-1}, c^{i^2j^2}, d^p\}$. Let H be the following EOL form

$$H:\quad S\to ab^{j-1},\quad a\to c^{i^2j^2},\quad b\to\Lambda,\quad c\to d^{p/i^2j^2},\quad d\to\Lambda.$$

One can easily see that $L(H)=(L(G)$ and comparing H and G one gets $\mathscr{L}(H)\subsetneq$

$\mathscr{L}(F)$. The fact that F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$. Let $F'' = (V'', \Sigma'', P'', S'')$.

Observe that each language of $\mathscr{L}(H)$ must contain at least three words (interpretations of ab^{j-1}, $c^{i^2j^2}$ and d^p). Since F is initial forcing, the above observation and the choice of j imply that either

$$S'' \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} ab^{j-1} \underset{F''}{\overset{+}{\Rightarrow}} c^{i^2j^2} \underset{F''}{\overset{+}{\Rightarrow}} d^p, \quad \text{or} \quad S'' \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} ab^{j-1} \underset{F''}{\overset{+}{\Rightarrow}} d^p \underset{F''}{\overset{+}{\Rightarrow}} c^{i^2j^2}.$$

The latter is clearly impossible. Thus $c^{i^2j^2} \underset{F''}{\Rightarrow} d^p$. Then obviously $c \underset{F''}{\overset{+}{\Rightarrow}} d^{p/i^2j^2}$. But then the structure of F'' yields $c \underset{F''}{\overset{+}{\Rightarrow}} y$ where y is a terminal word, $1 < |y| \le \max\{ij, l\}$. Consequently $c^{i^2j^2} \underset{F''}{\overset{+}{\Rightarrow}} y^{i^2j^2} = w$, w a terminal word with $i^2j^2 < |w| < p$; a contradiction.

(2) $j = 1$ and $i > 1$. Then let G be the EOL form which is defined as follows.

$$\begin{aligned} G\colon\quad & S \to a, \quad a \to AB^{i-1}, \quad a \to AC^{i-1}, \quad A \to a, \quad B \to b, \quad C \to c, \\ & b \to \Lambda, \quad c \to \Lambda \quad \text{if } a \underset{P}{\to} \Lambda, \quad \text{and} \\ & b \to D^i, \quad c \to D^i, \quad D \to \Lambda \quad \text{if } a \to \Lambda \text{ does not belong to } P. \end{aligned}$$

Obviously $G \lhd F$ and $L(G) = \{a, ab^{i-1}, ac^{i-1}\}$. Let H be the EOL form defined as follows.

$$H\colon\quad S \to ab^{i-1}, \quad a \to a, \quad b \to c, \quad c \to \Lambda.$$

One can easily see, comparing G and H, that $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. The fact that F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'') = L(H)$. Let $F'' = (V'', \Sigma'', P'', S'')$.

Observe that each language of $\mathscr{L}(H)$ contains at least three words. The structure of F'' implies either

$$S'' \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} a \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} ab^{i-1} \quad \text{or} \quad S'' \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} a \underset{\text{cnt}\, F''}{\overset{+}{\Rightarrow}} ac^{i-1}.$$

But then using isolation techniques one can prove that $\{a, ab^{i-1}\}$ or $\{a, ac^{i-1}\}$ belongs to $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$; a contradiction.

(3) $i = j = 1$ and $l > 1$. Then let G be the EOL form which is defined as follows.

$$\begin{aligned} G\colon\quad & S \to a, \quad a \to ab^{l-1}, \quad a \to ac^{l-1}, \quad b \to \Lambda, \\ & c \to \Lambda, \quad \text{if } a \underset{P}{\to} \Lambda, \quad \text{and} \\ & b \to A, \quad c \to A, \quad A \to \Lambda \quad \text{if } a \to \Lambda \text{ does not belong to } P. \end{aligned}$$

Obviously $G \lhd F$ and $L(G) = \{a, ab^{l-1}, ac^{l-1}\}$. Let H be the EOL form which is defined as follows.

$$H\colon\quad S \to ab^{l-1}, \quad a \to a, \quad b \to c, \quad c \to \Lambda.$$

Then proceeding as in (2) one can easily derive a contradiction.

Since (1), (2) and (3) exhaust all possible cases and for each of them the assumption F is good leads to a contradiction, F must be bad. Hence the lemma holds. □

Next we consider the cases where one of $\{i, j, l\}$ equals zero or is undefined (note that $L(F) \neq \emptyset$ yields $j \geq 1$ in the case considered) and the product of the remaining two parameters exceeds 1. First of all we investigate the case where $S \underset{P}{\rightarrow} S$, $i=1$ and $j>1$. This is done in Lemma 4.4. Then in Lemmas 4.5 and 4.6 the general situations are treated.

Lemma 4.4 *Let $F=(\{S,a\},\{a\},P,S)$ be a terminal erasing initial forcing* EOL *form such that $L(F) \neq \emptyset$ and $a \underset{P}{\rightarrow} x$ implies $x \in a^* \cup S^*$. Moreover let $S \underset{P}{\rightarrow} S$, $i=1$, $j>1$ and ($l=0$ or l is undefined). Then F is bad.*

Proof. Let F, i, j and l be as in the statement of the lemma. Let $D_{a,1}$ and $D_{a,2}$ be the following infinite derivations.

$$D_{a,1}\colon \quad a \underset{F}{\Rightarrow} x_1 = S \underset{F}{\Rightarrow} x_2 = a^j \underset{F}{\Rightarrow} x_3 = S^j \underset{F}{\Rightarrow} x_4 = a^{j^2} \underset{F}{\Rightarrow} \cdots.$$

$$D_{a,2}\colon \quad a \underset{F}{\Rightarrow} y_1 = S \underset{F}{\Rightarrow} y_2 = S \underset{F}{\Rightarrow} y_3 = S \underset{F}{\Rightarrow} y_4 = S \underset{F}{\Rightarrow} \cdots.$$

Let $m \geq 2$ be an integer such that $\#_a x_{2m} > j^3$. Then if $S \underset{P}{\rightarrow} \Lambda$, let

$$D_{a,1,1}\colon \quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m} = a^{j^m} \underset{F}{\Rightarrow} S^{j^m} \underset{F}{\Rightarrow} a^{j^2};$$

$$D_{a,1,2}\colon \quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m+1} \underset{F}{\Rightarrow} a^{j^{m+1}};$$

$$D_{a,2,1}\colon \quad a \underset{F}{\Rightarrow} y_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} y_{2m+1} = S \underset{F}{\Rightarrow} \Lambda.$$

If $a \underset{P}{\rightarrow} \Lambda$, let

$$D_{a,1,1}\colon \quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m} = a^{j^m} \underset{F}{\Rightarrow} S^j \underset{F}{\Rightarrow} a^{j^2};$$

$$D_{a,1,2}\colon \quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} x_{2m+1} \underset{F}{\Rightarrow} a^{j^{m+1}};$$

$$D_{a,2,1}\colon \quad a \underset{F}{\Rightarrow} x_1 \underset{F}{\Rightarrow} \cdots \underset{F}{\Rightarrow} y_{2m} = S \underset{F}{\Rightarrow} a^j \underset{F}{\Rightarrow} \Lambda.$$

Furthermore let

$$D_S\colon \quad S \underset{F}{\Rightarrow} a^j \quad \text{and} \quad D_{a,\Lambda}\colon \quad a \overset{\leq 2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} \Lambda.$$

Let μ be the dfl-substitution on $\{S,a\}^*$ defined by $\mu(S)=\{S\}$ and $\mu(a)=\{a,b,c,d\}$. Let $p=j^{m+1}$,

$$E_1\colon \quad S \Rightarrow ab^{j-1}, \qquad E_2\colon \quad a \overset{+}{\Rightarrow} c^{j^2}, \qquad E_3\colon \quad a \overset{+}{\Rightarrow} d^p,$$

$$E_4\colon \quad b \overset{+}{\Rightarrow} \Lambda, \qquad E_5\colon \quad c \overset{+}{\Rightarrow} \Lambda, \qquad E_6\colon \quad d \overset{+}{\Rightarrow} \Lambda$$

be such that $(E_1,E_2,E_3,E_4,E_5,E_6)$ is an isolated derivation tuple of $\tau=(D_S, D_{a,1,1},$

$D_{a1,2}, D_{a,2,1}, D_{a,\Lambda}, D_{a,\Lambda}$) modulo μ; let G be the EOL form based on τ and S. Clearly $G \lhd F$ and a close inspection of E_1 through E_6 yields $L(G)=\{ab^{j-1}, c^{j^2}, d^p\}$. Let H be the following EOL form.

$$H:\quad S \to ab^{j-1}, \quad a \to c^{j^2}, \quad b \to \Lambda, \quad c \to d^{p/j^2}, \quad d \to \Lambda.$$

One can easily see that $L(H)=L(G)$ and comparing H and G one gets $\mathscr{L}(H) \subseteq \mathscr{L}(F)$. The fact that F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'')=L(H)$. Let $F''=(V'', \Sigma'', P'', S'')$.

Observe that each language of $\mathscr{L}(H)$ must contain at least three words (interpretations of ab^{j-1}, c^{j^2} and d^p). Since F is initial forcing, the above observation and the choice of j imply that either

$$S'' \overset{+}{\underset{\mathrm{cnt}\,F''}{\Rightarrow}} ab^{j-1} \overset{+}{\underset{F''}{\Rightarrow}} c^{j^2} \overset{+}{\underset{F''}{\Rightarrow}} d^p \quad \text{or} \quad S'' \overset{+}{\underset{\mathrm{cnt}\,F''}{\Rightarrow}} ab^{j-1} \overset{+}{\underset{F''}{\Rightarrow}} d^p \overset{+}{\underset{F''}{\Rightarrow}} c^{j^2}.$$

The latter is clearly impossible. Thus $c^{j^2} \overset{+}{\underset{F''}{\Rightarrow}} d^p$. Then obviously $c \overset{+}{\underset{F''}{\Rightarrow}} d^{p/j^2}$. But then the structure of F'' yields $c \overset{+}{\underset{F''}{\Rightarrow}} y$ where y is a terminal word and $1<|y|\le j$. Consequently $c^{j^2} \overset{+}{\underset{F''}{\Rightarrow}} y^{j^2}=w$, w a terminal word with $j^2<|w|<p$; a contradiction. Hence F must be bad. $\square$

Lemma 4.5. *Let $F=(\{S,a\},\{a\},P,S)$ be a termimal erasing initial forcing* EOL *form such that $L(F)\neq\emptyset$ and $a \underset{P}{\to} x$ implies $x \in a^* \cup S^*$. Moreover, if $i\ge 2$, then $S \to S$ does not belong to P. Assume that ($l=0$ or l is undefined) and $ij>1$. Then F is bad.*

Proof. Let F, i, j and l be as in the statement of the lemma. If $S \underset{P}{\to} S$, then $i=1$, $j>1$ and thus by Lemma 4.4, F is bad. Therefore assume $S \to S$ does not belong to P.

One can easily see that F has the following property.

(4.2) Let $S \overset{n}{\underset{F}{\Rightarrow}} w$. If n is odd, then $w \in a^*$ and if n is even, then $w \in S^*$.

Recall that $a \overset{\le 2}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} \Lambda$ because F is terminal erasing. We consider now several cases separately.

(i.1) $j\ge 4$. Let H be the following EOL form.

$$H:\quad S \to aba^{j-2}, \quad a \to \Lambda, \quad b \to ccb^{j-2}, \quad c \to N, \quad N \to N.$$

Obviously $L(H)=\{aba^{j-2}, ccb^{j-2}\}$, $\mathscr{L}(H) \subseteq \mathscr{L}(F)$ and each language of $\mathscr{L}(H)$ contains at least two different words. That F is bad is now proved by contradiction. Assume that F is good. Then there exist $F'' \lhd F' \lhd F$ such that $\mathscr{L}(F'') \subseteq \mathscr{L}(F') = \mathscr{L}(H)$ and $L(F'')=L(H)$. Let $F''=(V'', \Sigma'', P'', S'')$. We have

$$S'' \underset{P''}{\to} aba^{j-2} \quad \text{or} \quad S'' \underset{P''}{\to} ccb^{j-2}.$$

Let Z be a new symbol and let $\bar{P}$ result from P'' by fixing for every $\alpha \in V''$ an α-production $\alpha \to x_\alpha$ from P''. Then the EOL form $\bar{F}$ is defined as follows.

$$\bar{F} = (V'' \cup \{Z\}, \Sigma'', \bar{P} \cup \tilde{P}, Z)$$

with $\tilde{P} = \{Z \to aba^{j-2}\}$ if $S \underset{P}{\Rightarrow} aba^{j-2}$ and $\tilde{P} = \{Z \to ccb^{j-2}\}$ otherwise. Clearly $\bar{F} \lhd F$, $\emptyset \neq L(\bar{F}) \subseteq L(F'')$ and $L(\bar{F}) \in \mathscr{L}(H)$. Since $L(\bar{F})$ must contain at least two different words, $L(\bar{F}) = L(H)$ holds. Thus $Z \underset{\bar{F}}{\Rightarrow} u \overset{2}{\underset{\bar{F}}{\Rightarrow}} v$ and either

(a) $u = aba^{j-2}$, $v = ccb^{j-2}$, or

(b) $u = ccb^{j-2}$, $v = aba^{j-2}$.

For a letter α of the total alphabet of $\bar{F}$ let y_α be such that $\alpha \overset{2}{\underset{\bar{F}}{\Rightarrow}} y_\alpha$. Observe that y_α is uniquely defined since $\bar{F}$ is deterministic.

If (a) holds, then $ccb^{j-2} = y_a y_b y_a^{j-2}$. $y_a \neq \Lambda$ is clearly impossible (for then $c \in \operatorname{alph} y_a$ and $y_a y_b y_a^{j-2}$ contains at least $j-1 \geq 3$ occurrences of c). Thus $y_a = \Lambda$, $y_b = ccb^{j-2}$. Then

$$Z \overset{+}{\underset{\bar{F}}{\Rightarrow}} ccb^{j-2} \overset{2}{\underset{\bar{F}}{\Rightarrow}} y_c y_c y_b^{j-2}$$

and from $\bar{F} \lhd F$ and (4.2) we get $y_c y_c y_b^{j-2}$ is a terminal word. But $y_c y_c y_b^{j-2}$ contains at least $2(j-2) \geq 4$ occurrences of c; a contradiction.

If (b) holds, then $aba^{j-2} = y_c y_c y_b^{j-2}$. Each of the following statements is clearly impossible:

$$b \in \operatorname{alph} y_c, \qquad b \in \operatorname{alph} y_b.$$

Hence a contradiction easily follows. Consequently F is bad.

(i.2) $j = 1$ and $i \geq 4$. Then let H be the following EOL form.

$$H\colon\ S \to d, \quad d \to aba^{i-2}, \quad a \to \Lambda, \quad b \to ccd^{i-2}, \quad c \to N, \quad N \to N.$$

Now $L(H) = \{d, aba^{i-2}, ccd^{i-2}\}$. An analogous reasoning as under (i.1) then yields the badness of F.

(ii.1) $j = 3$. Let H be the following EOL form.

$$H\colon\ S \to aab, \quad a \to b, \quad b \to c, \quad c \to d, \quad d \to N, \quad N \to N.$$

Obviously $L(H) = \{aab, bbc, ccd\}$, $\mathscr{L}(H) \subseteq \mathscr{L}(F)$ and each language of $\mathscr{L}(H)$ contains at least three different words. That F is bad is now proved by contradiction. Assume that F is good. Then F', F'' and $\bar{F}$ are constructed analogous to the construction under (i.1). Observe that $\bar{F}$ is deterministic and $L(\bar{F}) = L(H)$. We then have

$$Z \underset{\bar{F}}{\Rightarrow} u \overset{2}{\underset{\operatorname{cnt}\bar{F}}{\Rightarrow}} v \overset{2}{\underset{\operatorname{cnt}\bar{F}}{\Rightarrow}} w$$

such that one of the following must be the case.

(a) $u = aab$, $v = bbc$, $w = ccd$.

(b) $u = aab$, $v = ccd$, $w = bbc$.

(c) $u = bbc$, $v = aab$, $w = ccd$.

(d) $u = bbc$, $v = ccd$, $w = aab$.
(e) $u = ccd$, $v = aab$, $w = bbc$.
(f) $u = ccd$, $v = bbc$, $w = aab$.

For a letter α of the total alphabet of $\bar{F}$ let y_α be such that $\alpha \overset{2}{\underset{\bar{F}}{\Rightarrow}} y_\alpha$. Observe that y_α is uniquely defined since $\bar{F}$ is deterministic. Observe that

(4.3) for $\alpha \in \{a, b, c\}$, $|y_\alpha| \leq 1$,

since otherwise (using $\bar{F} \lhd F$ and (4.2)) a terminal word of length greater than or equal to four would be in $L(\bar{F})$.

For each of the cases (a) through (f) we will now derive a contradiction.

If (a) or (d) holds, then (4.3) yields $y_c = d$, hence $Z \overset{+}{\underset{\bar{F}}{\Rightarrow}} ccd \overset{2}{\underset{\bar{F}}{\Rightarrow}} ddy_d$. From $\bar{F} \lhd F$ and (4.2) we conclude ddy_d is a terminal word; a contradiction.

If (c) or (f) holds, then (4.3) yields $y_b = a$, hence $Z \overset{+}{\underset{\bar{F}}{\Rightarrow}} aab \overset{2}{\underset{\bar{F}}{\Rightarrow}} y_a y_a a$. From $\bar{F} \lhd F$ and (4.2) we conclude $y_a y_a a$ is a terminal word; a contradiction.

If (e) holds, then (4.3) yields $y_b = c$, hence $Z \overset{+}{\underset{\bar{F}}{\Rightarrow}} bbc \overset{2}{\underset{\bar{F}}{\Rightarrow}} ccy_c$. From $\bar{F} \lhd F$ and (4.2) we conclude ccy_c is a terminal word. This is only possible if $y_c = d$. Then $Z \overset{+}{\underset{\bar{F}}{\Rightarrow}} ccd \overset{2}{\underset{\bar{F}}{\Rightarrow}} ddy_d$. Again using $\bar{F} \lhd F$ and (4.2) we conclude ddy_d is a terminal word; a contradiction.

Since we have derived a contradiction in the cases (a) through (f), F must be bad.

(ii.2) $j = 1$ and $i = 3$. Then let H be the following EOL form.

$$H\colon\quad S \to e, \quad e \to aab, \quad a \to b, \quad b \to c, \quad c \to d, \quad d \to N, \quad N \to N.$$

Now $L(H) = \{e, aab, bbc, ccd\}$. An analogous reasoning as under (ii.1) then yields the badness of F.

(iii.1) $j = 2$. Then let H be the following EOL form.

$$H\colon\quad S \to ab, \quad a \to A, \quad A \to d, \quad b \to c, \quad c \to b, \quad d \to a.$$

Obviously $L(H) = \{ab, ac, db, dc\}$, $\mathscr{L}(H) \subseteq \mathscr{L}(F)$ and each language of $\mathscr{L}(H)$ contains at least four different words. That F is bad is now proved by contradiction. Assume that F is good. Then F', F'' and $\bar{F}$ are constructed analogous to (i.1). Observe that $\bar{F} \lhd F$, $\bar{F}$ is deterministic and $L(\bar{F}) = L(H)$. We then have

$$Z \underset{\bar{F}}{\Rightarrow} w_1 \overset{2}{\underset{\mathrm{cnt}\,\bar{F}}{\Rightarrow}} w_2 \overset{2}{\underset{\mathrm{cnt}\,\bar{F}}{\Rightarrow}} w_3 \overset{2}{\underset{\mathrm{cnt}\,\bar{F}}{\Rightarrow}} w_4 \quad \text{where } \{w_1, w_2, w_3, w_4\} = L(H)$$

Inspecting $L(H)$, without loss of generality we can assume that $w_1 = ab$ (The other cases follow by symmetry). Then one of the following must be the case

(a) $w_2 = ac$, $w_3 = db$, $w_4 = dc$.
(b) $w_2 = ac$, $w_3 = dc$, $w_4 = db$.
(c) $w_2 = db$, $w_3 = ac$, $w_4 = dc$.
(d) $w_2 = db$, $w_3 = dc$, $w_4 = ac$.
(e) $w_2 = dc$, $w_3 = ac$, $w_4 = db$.
(f) $w_2 = dc$, $w_3 = db$, $w_4 = ac$.

For a letter of the total alphabet of $\bar{F}$ let y_α be such that $\alpha \overset{2}{\underset{\bar{F}}{\Rightarrow}} y_\alpha$. Observe that y_α is uniquely defined since $\bar{F}$ is deterministic. Also observe that

(4.4) If $\alpha\beta \in L(H)$, $\alpha, \beta \in \{a, b, c, d\}$ and $|y_\alpha| = 2$, then $|y_\beta| = 0$.

This follows from (4.2), $\bar{F} \lhd F$ and the fact that $L(H)$ contains only words of length two.

For each of the cases (a) through (f) we will now derive a contradiction. In cases (a) and (b), $y_a \notin \{a, ac\}$ since otherwise neither w_3 nor w_4 can be derived. Thus $y_a = \Lambda$ and $y_b = ac$.

If (a) holds, then $y_a = \Lambda$, $y_b = ac$, $y_c = db$, $y_d = \Lambda$. Hence $w_4 = dc = ac$; a contradiction.

If (b) holds, then $y_a = \Lambda$, $y_b = ac$, $y_c = dc$, $y_d = \Lambda$. Hence $w_4 = db = dc$; a contradiction.

In cases (c) and (d), $y_b \notin \{b, db\}$. Thus $y_b = \Lambda$ and $y_a = db$.

If (c) holds, then $y_a = db$, $y_b = \Lambda$, $y_c = \Lambda$, $y_d = ac$. Hence $w_4 = dc = db$; a contradiction.

If (d) holds, then $y_a = db$, $y_b = \Lambda$, $y_c = \Lambda$, $y_d = dc$. Hence $w_4 = ac = dc$; a contradiction.

If (e) holds, then clearly $y_c \notin \{c, ac\}$. Thus $y_c = \Lambda$, $y_d = ac$, $y_a = db$, $y_b = \Lambda$. Hence $w_2 = dc = db$; a contradiction.

If (f) holds, then clearly $y_d \notin \{d, db\}$. Thus $y_c = db$, $y_d = \Lambda$, $y_a = \Lambda$, $y_b = ac$. Hence $w_2 = dc = ac$; a contradiction.

Since we have derived a contradiction in the cases (a) through (f), F must be bad.

(iii.2) $j = 1$ and $i = 2$. Then let H be the following EOL form.

$$H: \quad S \to e, \quad e \to ab, \quad a \to A, \quad A \to d, \quad b \to c, \quad c \to b, \quad d \to a.$$

An analogous reasoning as under (iii.1) then yields the badness of F. Since (i.1) through (iii.2) exhaust all possibilities for $ij > 1$ the lemma holds. □

Lemma 4.6. *Let $F = (\{S, a\}, \{a\}, P, S)$ be a terminal erasing initial forcing* EOL *form such that $L(F) \neq \emptyset$ and $a \underset{P}{\to} x$ implies $x \in a^* \cup S^*$. Assume that ($i = 0$ or i is undefined) and $jl > 1$. Then F is bad.*

Proof. Let F, i, j and l be as in the statement of the lemma. Observe that F has the following property.

(4.5) If $a \overset{*}{\underset{F}{\Rightarrow}} w$, then $w \in a^*$.

Also observe that for an interpretation G of F interpretations of the production $S \to S$ (if $S \underset{P}{\to} S$) can only occur in the initial piece of a derivation. Then reasoning as in Lemma 4.5 (i.1) through (iii.2) one gets that F is bad. The following modifications should be made.

(a) i should be replaced by l.

(b) Whenever we used productions $Z \to x$ we should use E: $Z \overset{+}{\Rightarrow} x$ an isolated derivation of D: $S'' \overset{+}{\underset{\mathrm{cnt}\,F}{\Rightarrow}} x$.

(c) Whenever we used $\overset{2}{\underset{\bar{F}}{\Rightarrow}}$ we should now use $\underset{\bar{F}}{\Rightarrow}$. □

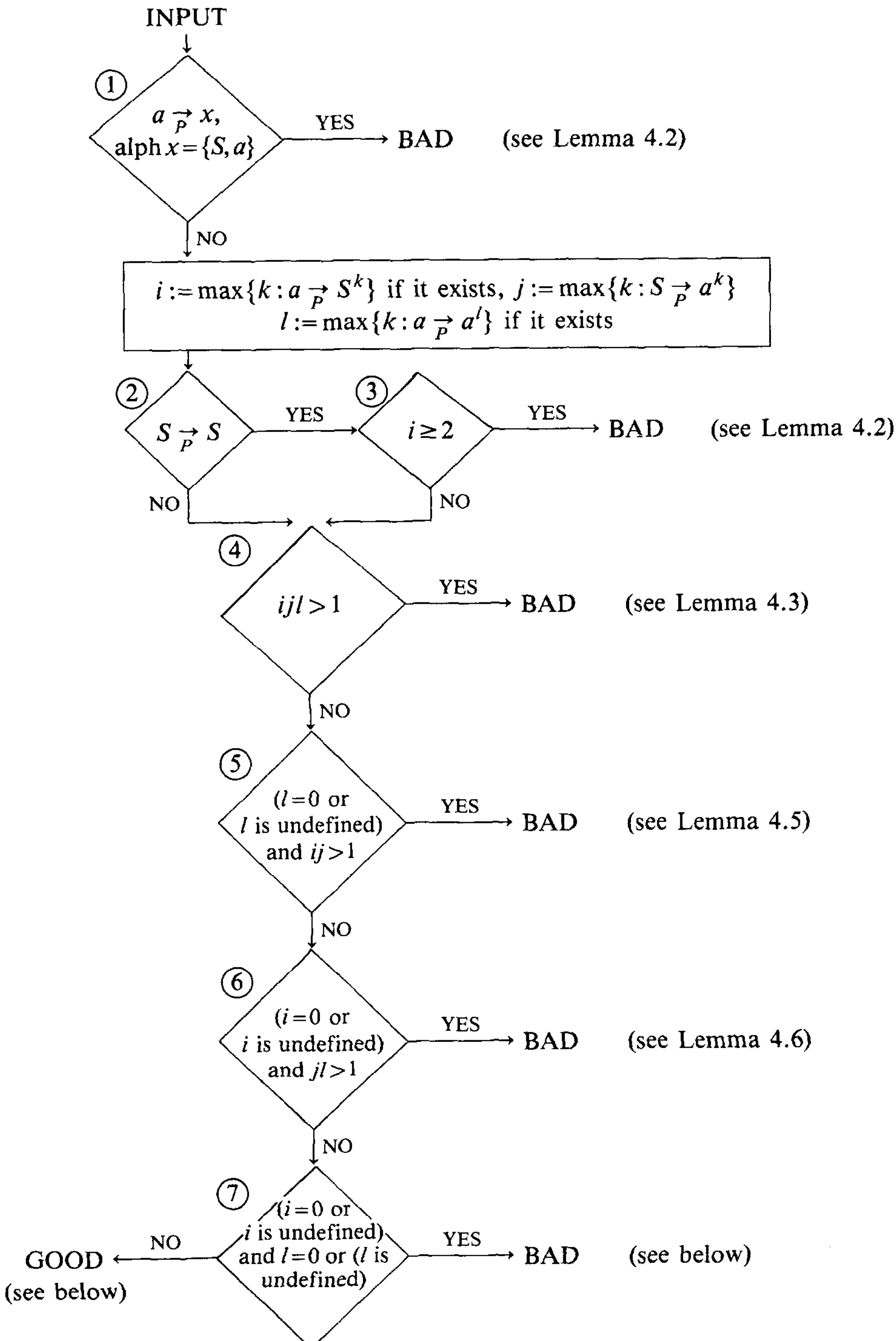
INPUT
1
$a \underset{P}{\to} x$, alph $x = \{S, a\}$
YES
BAD
(see Lemma 4.2)
NO
$i := \max\{k : a \underset{P}{\to} S^k\}$ if it exists, $j := \max\{k : S \underset{P}{\to} a^k\}$
$l := \max\{k : a \underset{P}{\to} a^l\}$ if it exists
2
$S \underset{P}{\to} S$
YES
3
$i \geq 2$
YES
BAD
(see Lemma 4.2)
NO
NO
4
$ijl > 1$
YES
BAD
(see Lemma 4.3)
NO
5
($l = 0$ or l is undefined) and $ij > 1$
YES
BAD
(see Lemma 4.5)
NO
6
($i = 0$ or i is undefined) and $jl > 1$
YES
BAD
(see Lemma 4.6)
NO
7
($i = 0$ or i is undefined) and $l = 0$ or (l is undefined)
YES
BAD
(see below)
NO
GOOD
(see below)

Diagram 3.

We are now able to state the main result of the section.

Theorem 4.1. *Let* $F = (\{S, a\}, \{a\}, P, S)$ *be a terminal erasing and initial forcing* EOL *form such that* $L(F) \neq \emptyset$. *Then it is decidable whether or not* F *is good.*

Proof. Let F be as in the statement of the theorem. Then to decide whether or not F is good, we use Diagram 3.

All tests involved are obviously effective. We only have to prove the validity of test 7. If the outcome of test 7 is positive, then the only a-production of F is $a \to \Lambda$. Consequently Lemma 1.1 yields that F is bad. If the outcome of test 7 is negative then we end with an EOL form $F = \{V, \Sigma, P, S)$ such that $L(F) = \{a\}$, $S \underset{P}{\to} a$ and ($a \underset{P}{\to} a$ or $a \underset{P}{\to} S$). Then one can easily prove that F is good (see, e.g., [12]). □

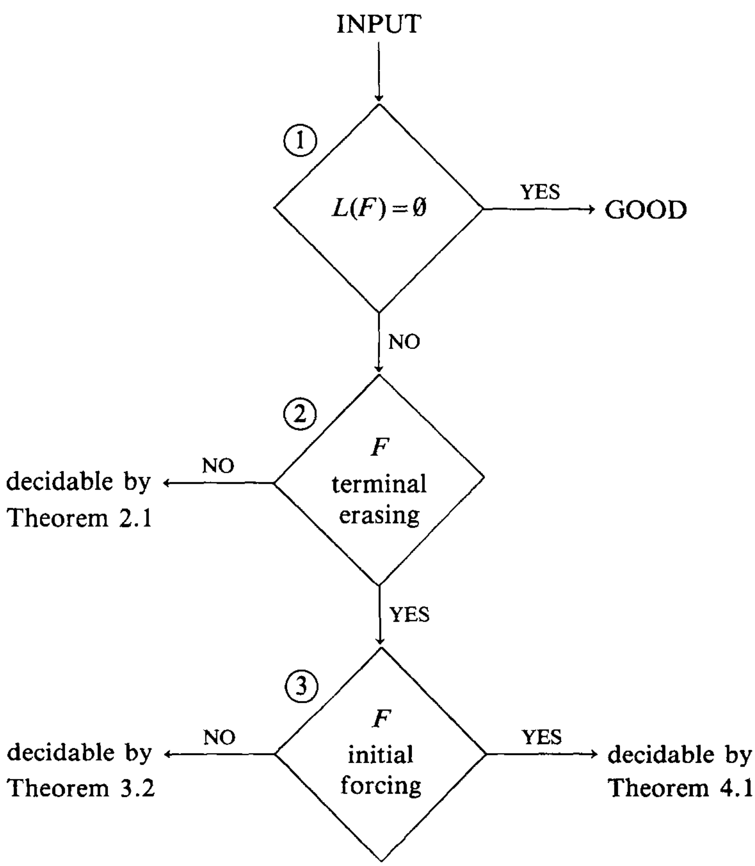

Diagram 4.

5. Main results

Using the results of the previous sections we are now able to prove that goodness of $\{S, a\}$-forms is decidable.

Theorem 5.1. *Let* $F = (\{S, a\}, \{a\}, S)$ *be an* EOL *form. Then it is decidable whether or not* F *is good.*

Proof. Let F be as in the statement of the theorem. Then to decide whether or not F is good we use Diagram 4.

Test 1 is clearly effective (see, e.g., [14]), test 2 is effective by Lemma 3.1 and test 3 is obviously effective. The theorem then follows from Diagram 4. □

As can be seen from the above sections, there exists only two 'different types' of good $\{S, a\}$-forms. We will need the following definition first.

Definition. Let F be an EOL form. Then the *quadratic language family* of F, denoted $\mathscr{L}^2(F)$, is defined by $\mathscr{L}^2(F) = \{\mathscr{L}(F) : F' \lhd F\}$. Furthermore

$$\mathscr{L}^2(\mathrm{EOL}) = \{\mathscr{L}(F) : F \text{ an EOL form}\}. \quad \square$$

We then have the following result.

Theorem 5.2. *Let* $F = (\{S, a\}, \{a\}, P, S)$ *be a good* EOL *form such that* $L(F) \neq \emptyset$. *Then* $\mathscr{L}^2(F) = \{\mathscr{L}^2(\mathrm{EOL})$ *or* $\mathscr{L}^2(F) = \{\mathscr{L}_n : n \geq 1\}$ *where* $\mathscr{L}_n = \{K : K = \{a_1, a_2, \ldots, a_m\}, m \geq n\}$.

Proof. Let F be as in the statement of the theorem. Recall that F is vomplete if and only if $\mathscr{L}^2(F) = \mathscr{L}^2(\mathrm{EOL})$ (see, e.g., [15]). Also recall that for the EOL form $F : S \to a$, $a \to a$ we have $\mathscr{L}^2(F) = \{\mathscr{L}_n : n \geq 1\}$ where $\mathscr{L}_n$ are defined as above (see, e.g., [12]). Then the theorem easily follows from the above observations and Theorems 2.1, 3.2, 4.1 and 5.1. □

In the literature most examples of good EOL forms are good $\{S, a\}$-forms or forms strongly connected to $\{S, a\}$-forms. The reason why only few 'different' examples are given is simple: as expressed by Theorem 5.2, only two 'different types' exist.

We end the paper by the remark that generalizing the above results to arbitrary EOL forms might be very difficult. E.g., if an EOL form F has more than one terminal symbol, then the property that $L(F)$ contains at most one word of each length does not hold any more.

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